A CANONICAL SYSTEM OF VARIABLES IN THE THEORY OF SATELLITE MOTION

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A method involving variation of arbitrary constants is used to solve the differetial equations for intermediate non-Kepler motion. A new system of canonical variables is introduced which is convenient for studying perturbations of this motion.

In this paper we will analyze the motion of a planet's satellite under the assumption that the main perturbing factor is the solar gravitational attraction, but that other smaller perturbations also operate.

Study of the motion of the satellites of planets in the solar system shows that in many cases the orbit of a satellite can be approximated quite accurately as a Kepler ellipse with a moving node and a rotating line of apsides, whose eccentricity and inclination undergo only slight periodic changes. The motion of such a satellite can be predicted theoretically quite successfully by using Kepler motion as an initial approximation. On the other hand, there are satellites whose osculating-orbit elements are so strongly affected by the sun that the motion cannot be approximated even crudely by a Kepler ellipse, e.g., Jupiter's eighth satellite [1-3]. An intermediate orbit of Kepler ellipses is not useful for theoretically predicting the motion of such satellites; a different intermediate orbit must be used as an initial approximation - one which already incorporates the basic effects of the solar attraction. In particular, one such intermediate orbit is that which we found previously [4, 5].

In this paper we treat this orbit as the unperturbed orbit. More precisely, we use as an unperturbed orbit that whose osculating elements are given the equations of [4, 5] which also take into account the secular and long-period solar perturbations.

In the usual form of the differential equations of perturbed motion in stellar mechanics, the right sides are airy sums of terms corresponding to the unperturbed and perturbed motion. In our case the differential equations of the unperturbed motion are found by transforming the equations of the Hill problem by the Zeipel method [6] and discarding terms of fourth or higher orders in $m = n_2/n_1$, where n_1 is the average motion of the satellite and n_2 is the average motion of the sun. Accordingly, to transform the initial differential equations of motion of a satellite into a form in which the terms corresponding to the unperturbed motion are clearly distinguished from those corresponding to the perturbing forces on the right side, they must transform the initial variables on the basis of the equations in [4, 5]. We assume below that these transformations have been carried out.

The problem which we take up below involves an application of a method of varying arbitrary constants to the differential equations, determining the perturbations of the intermediate orbit, and obtaining certain systems of variables which are convenient for the analysis. The corresponding mathematical operations are carried out by means of canonical transformations of the variables governing the satellite motion, since this is the simplest way to solve this problem. The initial system of canonical variables for the perturbed motion is found by solving the differential equations of the intermediate motion by the Hamilton-Jacobi method [7]. The characteristic function of the canonical system of differential equations for the intermediate motion incorporates corrections for the ellipticity of the solar orbit which we found in [8].

§1. INTEGRATION OF THE DIFFERENTIAL EQUATIONS OF INTERMEDIATE MOTION BY THE HAMILTON-JACOBI METHOD

We will write the differential equations of the intermediate motion of the satellite in terms of the Delaunay canonical variables L'', G'', H'', l'', g'', h'', over which the transformations indicated in [8] have been carried out. For simplicity we will denote these variables below as L, G, H, l, g, h. We will not use the variable pair Λ and λ introduced in [8], since they do not affect the nature of the intermediate motion.

We write the characteristic function of the intermediate motion (see §1 of /5/ and §3 of [8]) as

$$\Psi = \Psi(L, G, H, g) = -\frac{\mu^2}{2L^2} + \frac{1}{16}\gamma m^2 \frac{L^4}{\mu^2} \times$$
$$\times \{2(1-3c_2^2) - 2(1-c_2^2)c_1 + 3(1-\eta^2)[1 - 11c_1 -$$
$$- 5(1+c_1)c^2 - 5(1+c_1)(1-c^2)\cos 2g]\}, \tag{1}$$

where

$$\bar{\gamma} = \frac{m_2(1-e_2^2)^{-3/2}}{m_0+m_1+m_2}, \quad c_1 = \frac{9}{8}\bar{\gamma}m\frac{L^2H}{\mu^2}\left(1+\frac{2}{3}e_2^2\right),$$

$$m = \frac{n_2}{n_1}, \quad \mu = \frac{f(m_0+m_1)}{n_1^2}, \quad c_2 = \frac{H}{L},$$

$$\eta = \frac{G}{L}, \quad c = \frac{H}{G}, \tag{2}$$

m_0, m_1, and m_2 are the masses of the planet, satellite, and sun, respectively; e_2 is the eccentricity of the planet-centered orbit of the sun, and f is the gravitational constant.

Then the differential equations of the intermediate motion of the satellite are

$$\frac{dl}{d\tau} = \frac{\partial\Psi}{\partial L}, \quad \frac{dg}{d\tau} = \frac{\partial\Psi}{\partial G}, \quad \frac{dh}{d\tau} = \frac{\partial\Psi}{\partial H},$$

$$\frac{dL}{d\tau} = -\frac{\partial\Psi}{\partial l}, \quad \frac{dG}{d\tau} = -\frac{\partial\Psi}{\partial g}, \quad \frac{dH}{d\tau} = -\frac{\partial\Psi}{\partial h}, \tag{3}$$

where $\tau = n_1(t-t_0)$, t is the time, and t_0 is its initial value.

We will use the Hamilton-Jacobi method to solve system (3) (p. 310 of [7]). We denote by $W = W(\tau, l, g, h, \alpha_1, \alpha_2, \alpha_3)$ the total integral of the Hamilton-Jacobi equation. Since in our case characteristic function (1) of canonical system (3) does not depend on τ, l or h, we can write

$$W = \mathcal{E}\tau + \alpha_1 l + \alpha_3 h + W_1(g), \tag{4}$$

where $\mathcal{E}$, α_1 and α_3 are arbitrary constants. There are three such constants, precisely in accordance with the requirements of the Hamilton-Jacobi method. We must now determine the function $W_1(g)$.

Introducing the new arbitrary constant α_2 related to to $\mathcal{E}$ by

$$\mathcal{E} = \frac{\mu^2}{2\alpha_1^2} - \frac{1}{16}\bar{\gamma}m^2\frac{\alpha_1^4}{\mu^2}\left[2\left(1-3\frac{\alpha_3^2}{\alpha_1^2}\right)+3\alpha_2 - \frac{3}{4}\frac{\bar{\gamma}m\alpha_3}{\mu^2}(3+2e_2^2)\alpha_1^2 - \alpha_3^2)\right], \tag{5}$$

we find, after some straight forward calculations, that $W_1(g)$ must satisfy an equation of the form

$$\Phi\left(\frac{\partial W_1}{\partial g}, \alpha_1, \alpha_2, \alpha_3\right) = A\left(\frac{\partial W_1}{\partial g}\right)^4 - B\left(\frac{\partial W_1}{\partial g}\right)^2 - C = 0, \tag{6}$$

where A, B and C are linear functions in $\sin^2 g$ with coefficients which depend on the constants α_1, α_2, α_3. Expressing the derivative $\frac{\partial W_1}{\partial g}$ in (6) as an explicit function of g, we can write W_1 as

$$W_1 = \int_{g_0}^{g} \frac{\partial W_1}{\partial g}\, dg. \tag{7}$$

Serious problems arise when an attempt is made to directly evaluate this integral or to reduce it to standard integrals, so we will follow a different path for evaluating W_1. We change the integration variable in Eq. (7) by setting z = tg g. When Eq. (6) converts into an algebraic equation for $\frac{\partial W_1}{\partial g}$ and z, and the integrand on the right side of Eq. (7) becomes a rational function for these variables. Our problem has thus been reduced to one of evaluating an Abelian integral which, it turns out, we can easily convert into elliptical integrals, even without analyzing the associated algebraic curve. We use the replacement

$$\frac{\partial W_1}{\partial g} = \alpha_1\sqrt{\xi} \tag{8}$$

and adopt ξ as a new integration variable. Then we find from (6)

$$z^2 = \mathrm{tg}^2\, g = \frac{\xi(\delta\xi - \varepsilon)}{\alpha\xi^2 - 2\beta\xi + \gamma}, \tag{9}$$

where

$$\alpha = 6(1-c_1), \quad 2\beta = \alpha + \gamma - \alpha_2,$$

$$\gamma = 10(1+c_1)c_2^2, \quad \delta = 4(1+4c_1), \quad \varepsilon = \delta + \alpha_2,$$

$$c_1 = \frac{9}{8}\bar{\gamma}m\frac{\alpha_1^2\alpha_3}{\mu^2}, \quad c_2 = \frac{\alpha_3}{\alpha_1}. \tag{10}$$

If we replace in the equation $(\delta\xi-\varepsilon)(\alpha\xi^2-2\beta\xi+\gamma)=0$. the constant α_2 by the constant $c_3=-3\alpha_2+2(1-3c_2^2)-2c_1(1-c_2^2)$, this equation becomes exactly equal to Eq. (5.14) of /4/, whose roots have been studied in detail [4]. In particular, it follows from this study that the numerator on the right side of Eq. (9) can vanish at allowed values of ξ $(0<\xi<1)$ only if we have $\alpha_2<0$. We conclude that for $\alpha_2>0$ the intermediate orbits are "librational;" i.e., the pericenters of the corresponding osculating elliptical orbits do not undergo secular motion. In this paper we consider only nonlibrational intermediate orbits; i.e., we will assume that $\alpha_2<0$. In this case, we can use the results of 6 [4] to write

$$z^2=\frac{\delta\xi(\xi_2-\xi)}{\alpha(\xi_1-\xi)(\xi-\xi_3)}, \tag{11}$$

where $0<\xi_3\leqslant\xi\leqslant\xi_2<1<\xi_1$. (In case of librational orbits, ξ_1 and ξ_2 would be interchanged in this inequality.

To completely reconcile our equations with those of [4, 5], we must take the negative route of (11):

$$z=\operatorname{tg} g=-\sqrt{\frac{\delta}{\alpha}\,\frac{\xi(\xi_2-\xi)}{(\xi_1-\xi)(\xi-\xi_3)}}. \tag{12}$$

Assuming

$$\xi=\xi_3+(\xi_2-\xi_3)\operatorname{sn}^2 u, \tag{13}$$

where $\operatorname{sn} u$ is the elliptical Jacobi function with modulus $k=\sqrt{\frac{\xi_2-\xi_3}{\xi_1-\xi_3}}$, we find

$$W_1=\frac{\alpha_1}{\Delta}\int\limits_0^u\frac{\delta c_2^2(1-\xi)^2-\alpha_2(\xi^2-c_2^2)}{(1-\xi)(\xi-c_2^2)}\,du,\quad \Delta=\sqrt{\alpha\delta(\xi_1-\xi_3)}. \tag{14}$$

The partial derivatives of W_1 with respect to α_i $(i=1,2,3)$ can also be written as elliptic integrals: applying to Eq. (6) the rule for differentiating implicit functions, we find

$$\frac{\partial W_1}{\partial\alpha_i}=-\int\limits_{g_0}^{g}\frac{\partial\Phi}{\partial\alpha_i}\left[\frac{\partial\Phi}{\partial\left(\frac{\partial W_1}{\partial g}\right)}\right]^{-1}dg,\quad (i=1,2,3). \tag{15}$$

Hence, after all these operations, we find

$$\frac{\partial W_1}{\partial\alpha_1}=-\frac{1}{\Delta(1+c_1)}\int\limits_0^u\frac{\bar\alpha_1\xi^2+\bar\beta_1\xi+\bar\gamma_1}{1-\xi}\,du,$$
$$\frac{\partial W_1}{\partial\alpha_2}=\frac{\alpha_1 u}{2\Delta},$$
$$\frac{\partial W_1}{\partial\alpha_3}=-\frac{1}{2\Delta(1+c_1)}\int\limits_0^u\frac{\bar\alpha_2\xi^2+\bar\beta_2\xi+\bar\gamma_2}{\xi-c_2^2}\,du, \tag{16}$$

where

$$\bar\alpha_1=-12c_1,\ \bar\beta_1=\alpha_2+24c_1,\ \bar\gamma_1=-(12-\alpha_2)c_1,$$
$$\bar\alpha_2=12\frac{c_1}{c_2},\ \ \bar\beta_2=8c_2-(12-28c_2^2-\alpha_2)\frac{c_1}{c_2}+32c_2c_1^2,$$
$$\bar\gamma_2=-[8+2\alpha_2+(28+3\alpha_2)c_1+32c_1^2]c_2. \tag{17}$$

This ends the solution of canonical system (3) by the Hamilton-Jacobi method; the final results are

$$L=\alpha_1,\ \beta_1=\frac{\partial g}{\partial\alpha_1}\tau+l+\frac{\partial W_1}{\partial\alpha_1},$$
$$G=\alpha_1\sqrt{\xi},\ \beta_2=\frac{\partial g}{\partial\alpha_2}\tau+\frac{\partial W_1}{\partial\alpha_2},$$
$$H=\alpha_3,\ \beta_3=\frac{\partial g}{\partial\alpha_3}\tau+h+\frac{\partial W_1}{\partial\alpha_3}, \tag{18}$$

where β_1, β_2, and β_3 are three new arbitrary constants.

We can use Eqs. (12) and (18) to write L, G, H, l, g, h as explicit functions of τ.

§2. APPLICATION OF THE METHOD OF VARYING ARBITRARY CONSTANTS

We now consider a canonical system of the form (3) whose characteristic function F differs from (1) by a perturbation term $R(L, G, H, l, g, h)$: $F=\Psi-R$, and we will use the method of varying arbitrary constants to solve Eqs. (18) for the unperturbed motion, treating α_i and β_i (i = 1, 2, 3) as unknown functions of the time. The functions α_i and β_i form a canonical system of variables which solve the following differential equations (p. 317 of [7]):

$$\frac{d\alpha_i}{d\tau} = \frac{\partial R}{\partial \beta_i}, \quad \frac{d\beta_i}{d\tau} = -\frac{\partial R_i}{\partial \alpha_i}, \ (i = 1, 2, 3). \tag{19}$$

The problem of applying the method of varying arbitrary constants to the general solution of canonical systems (3) has thus been solved. However, Eqs. (19) are in practice inconvenient, since, when the function R is differentiated with respect to α_i $(i=1, 2, 3)$, the independent variable τ appears as a factor in certain terms in the derivatives. To avoid this problem, we introduce the new canonical variables $L_1, L_2, L_3, l_1, l_2, l_3$, setting

$$\begin{aligned} l_1 &= \beta_1 - \frac{2I_1}{\alpha_1 K(1+c_1)}\left(\beta_2 - \frac{\partial \mathcal{E}}{\partial \alpha_2}\tau\right) - \frac{\partial \mathcal{E}}{\partial \alpha_1}\tau, \\ l_2 &= \frac{\pi\Delta}{\alpha_1 K}\left(\beta_2 - \frac{\partial \mathcal{E}}{\partial \alpha_2}\tau\right), \\ l_3 &= \beta_3 - \frac{I_3}{\alpha_1 K(1+c_1)}\left(\beta_2 - \frac{\partial \mathcal{E}}{\partial \alpha_2}\tau\right) - \frac{\partial \mathcal{E}}{\partial \alpha_3}\tau, \end{aligned} \tag{20}$$

where K is the fourth period of the function sn u, $\mathcal{E}$ is defined by Eq. (5), and

$$I_1 = -\int_0^K \frac{\bar{\alpha}_1\xi^2 + \bar{\beta}_1\xi + \bar{\gamma}_1}{1-\xi}\,du, \quad I_2 = -\int_0^K \frac{\bar{\alpha}_2\xi^2 + \bar{\beta}_2\xi + \gamma_2}{\xi - c_2^2}\,du.$$

The problem is to choose canonical variables L_1, L_2, and L_3 which are conjugate to l_1, l_2 and l_3. To do this, we find the function $S=S(\tau, \alpha_1, \alpha_2, \alpha_3, l_1, l_2, l_3)$, which satisfies the conditions

$$\frac{\partial S}{\partial \alpha_i} = \beta_i, \quad (i = 1, 2, 3).$$

We first take note of the following circumstance. Equation (14) gives W_1 as a function of the four quantities $u, \alpha_1, \alpha_2\ \alpha_3$: $W_1 = W_1(u, \alpha_1, \alpha_2, \alpha_3)$. Assuming $u=K=K(\alpha_1, \alpha_2, \alpha_3)$ and differentiating the function of three parameters $W(K, \alpha_1, \alpha_2, \alpha_3) = \overline{W}_1(\alpha_1, \alpha_2, \alpha_3)$ obtained thereby with respect to its argument, we find

$$\frac{\partial \overline{W}_1}{\partial \alpha_1} = \frac{I_1}{\Delta(1+c_1)}, \quad \frac{\partial \overline{W}_1}{\partial \alpha_2} = \frac{\alpha_1 K}{2\Delta}, \quad \frac{\partial \overline{W}_1}{\partial \alpha_3} = \frac{I_3}{2\Delta(1+c_1)}. \tag{21}$$

Now determining β_1, β_2, and β_3 from Eqs. (20) and using (21), we find

$$\frac{\partial S}{\partial \alpha_i} = \frac{\partial}{\partial \alpha_i}\left(\alpha_1 l_1 + \frac{2}{\pi}\overline{W}_1 l_2 + \alpha_3 l_3 + \mathcal{E}\tau\right), \quad (i = 1, 2, 3),$$

from which we find

$$S = \alpha_1 l_1 + \frac{2}{\pi}\overline{W}_1 l_2 + \alpha_3 l_3 + \mathcal{E}\tau.$$

According to the general theory (pp. 300-305 [7]), we find the canonical variables conjugate to variables (20),

$$L_1 = \alpha_1, \quad L_2 = \frac{2}{\pi}\overline{W}_1, \quad L_3 = \alpha_3,$$

and the new characteristic function

$$\bar{F} = \mathcal{E} + R. \tag{22}$$

The corresponding canonical system of equations is

$$\frac{dL_i}{d\tau} = \frac{\partial F}{\partial l_i}, \quad \frac{dl_i}{d\tau} = -\frac{\partial F}{\partial L_i}, \quad (i = 1, 2, 3). \tag{23}$$

CONCLUSION

We have found a system of canonical variables $L_i, l_i (i=1, 2, 3)$, which has the following properties: a) In the absence of the perturbation R in Hamiltonian (22), this system converts into the system of the variables for intermediate motion. b) The unperturbed component $\mathcal{E}$ of the Hamiltonian depends only on the variables L_i; it independent of τ and the angular variables l_i. c) The derivatives of perturbed function R_i with respect to the variables L_i and l_i contain the angular variables l_i and the independent variable τ only within the arguments of trigonometric functions.

In terms of the variables L_i, l_i, therefore, the differential of the motions of a satellite assume the usual form of the differential equations of perturbed motion for classical stellar mechanics. The classical methods of perturbation theory can thus be applied to Eqs. (13).

The complex analytical structure of L_i, l_i is not a serious disadvantage, since, whether we use these variables or some classical variables, we are forced to resort series expansions or to other approximate methods.

On the other hand, the variables L_i, l_i have a clear advantage over classical variables in problems involving the motion of stellar objects such as the outer satellites of Jupiter, since they determine the perturbations in that intermediate orbit which already reflects at least some of the most important solar perturbations.

In conclusion we note that, where convenient, it is easy to transform from Eqs. (13) to equations for variables (canonical or noncanonical) chosen in some different manner. For example, we could easily obtain a system of differential equations giving the parameters α_1, α_2 and α_3 and the angular variables l_i $(i=1, 2, 3)$ which appear in the equations of the intermediate motion (we omit these equations for brevity).

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