SOME EXACT SOLUTIONS OF THE DIRAC EQUATION

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Variables are separated in the Dirac equation for an electron moving in a uniform magnetic field and in a parallel, time-dependent electric field. Explicit solutions are found for certain particular fields.

GENERAL ANALYSIS OF THE PROBLEM

We consider the motion of an electron (of charge − e and rest mass m) in an external electromagnetic field having the potentials

$$A_x = -y_0 H_0 \Phi\left(\frac{y}{y_0}\right), \quad A_y = 0, \; A_z = -\frac{cE_0}{\alpha}\varphi(\alpha t), \; V = 0, \tag{1}$$

where y_0, α, H_0, and E_0 are certain dimensional constants, and $\Phi\left(\frac{y}{y_0}\right)$ и $\varphi(\alpha t)$ are dimensionless functions. The magnetic and electric fields for this potential are

$$H_x = H_y = E_x = E_y = 0, \; H_z = y_0 H_0 \frac{\partial\Phi}{\partial y} = H_0\Phi',$$
$$E_z = \frac{E_0}{\alpha}\frac{\partial\varphi}{\partial t} = E_0\varphi'. \tag{2}$$

The fields are thus parallel: they consist of a uniform and constant magnetic field and a uniform electric field which is not constant.

The electron wave function must satisfy the Dirac equation

$$i\hbar\frac{\partial}{\partial t}\Psi = \widehat{\mathscr{H}}\Psi, \; \widehat{\mathscr{H}} = c(\vec{\alpha}\vec{P}) + \rho_3 mc^2, \tag{3}$$

where $\vec{P} = -i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}$ is the kinetic momentum and $\vec{\alpha}$ and ρ_3 are Dirac matrices.

In this problem the integrals of motion are the quasimomenta p_1 and p_3, for the operators $\widehat{p}_1 = -i\hbar\frac{\partial}{\partial x}$ and $\widehat{p}_3 = -i\hbar\frac{\partial}{\partial z}$ connect with the Hamiltonian. Requiring the wave function to be an eigenfunction for these operators,

$$\widehat{p}_1\Psi = \hbar k_1\Psi \text{ and } \widehat{p}_3\Psi = \hbar k_3\Psi, \tag{4}$$

we seek a solution of the general problem in the form

$$\Psi(\vec{r}, t) = \frac{1}{L}e^{i(k_1x + k_3z)}, \quad \Psi(y, t), \tag{5}$$

where the four-component spinor Ψ(y, t) satisfies

$$\left(\frac{i}{c}\frac{\partial}{\partial t} - k_0\right)\Psi_{1,2} - \left(k_1 - \gamma y_0\Phi \mp \frac{\partial}{\partial y}\right)\Psi_{4,3} \mp \left(k_3 - \frac{eE_0}{\alpha\hbar}\varphi\right)\Psi_{3,4} = 0,$$
$$\left(\frac{i}{c}\frac{\partial}{\partial t} + k_0\right)\Psi_{3,4} - \left(k_1 - \gamma y_0\Phi \mp \frac{\partial}{\partial y}\right)\Psi_{2,1} \mp \left(k_3 - \frac{eE_0}{\alpha\hbar}\varphi\right)\Psi_{1,2} = 0, \tag{6}$$

where $k_0 = \frac{mc}{\hbar}$ and $\gamma = \frac{eH_0}{c\hbar}$. In this case there exists a spin integral motion (see also /1/), the operator

$$\mathscr{L} = \Pi_3\cos\theta + \Phi_3\sin\theta,$$

$$\vec{\Pi} = mc\vec{\sigma} + \rho_2[\vec{\sigma}\vec{P}], \; \Phi = -\rho_3[\vec{\sigma}\vec{P}], \tag{7}$$

where θ is the angle between the spin direction and the z axis. Requiring that wave function (5) be an eigenfunction of the operator $\mathscr{L}$,

$$\widehat{\mathscr{L}}\Psi = \zeta\hbar\mathscr{L}\psi, \tag{8}$$

($\zeta = \pm 1$ characterizes the two possible spin orientations, and L is the eigenvalue of the operator), we find that the spinor ψ must satisfy in addition to Eq. (6) the equations

$$(k_0 \cos\theta \mp \zeta\mathscr{L})\,\psi_{1,2} + \cos\theta\left(k_1 - \gamma y_0 \Phi \mp \frac{\partial}{\partial y}\right)\psi_{4,3} -$$

$$- i \sin\theta\left(k_1 - \gamma y_0 \Phi \mp \frac{\partial}{\partial y}\right)\psi_{2,1} = 0,$$

$$(k_0 \cos\theta \mp \zeta\mathscr{L})\,\psi_{3,4} - \cos\theta\left(k_1 - \gamma y_0 \Phi \mp \frac{\partial}{\partial y}\right)\psi_{2,1} +$$

$$+ i \sin\theta\left(k_1 - \gamma y_0 \Phi \mp \frac{\partial}{\partial y}\right)\psi_{4,3} = 0. \tag{9}$$

Combining these systems of equations, we can satisfy them simultaneously by choosing the spinor $\psi(y, t)$ in the following manner:

$$\psi_{1,3} = f(y)\,[\pm C_1\chi_1(t) + C_2\chi_2(t)],$$

$$\psi_{2,4} = g(y)\,[C_3\chi_1(t) \mp C_4\chi_2(t)]. \tag{10}$$

Here the functions f and g depend only on y and satisfy a system of two first-order equations,

$$\left\{\frac{d}{dy} - \gamma y_0\,\Phi\left(\frac{y}{y_0}\right) + k_1\right\} f(y) = -\lambda g(y),$$

$$\left\{\frac{d}{dy} + \gamma y_0\,\Phi\left(\frac{y}{y_0}\right) - k_1\right\} g(y) = \lambda f(y), \tag{11}$$

while the functions χ depend only on t,

$$\left\{\frac{i}{c}\frac{d}{dt} - \frac{eE_0}{\alpha\hbar}\,\varphi(\alpha t) + k_3\right\}\chi_1(t) = (K + k_3)\,\chi_2(t),$$

$$\left\{\frac{i}{c}\frac{d}{dt} + \frac{eE_0}{\alpha\hbar}\,\varphi(\alpha t) - k_3\right\}\chi_2(t) = (K - k_3)\,\chi_1(t), \tag{12}$$

where λ and K are numbers which are not yet known.

Choosing spinor $\psi(y, t)$ in form (10), we convert the systems (6) and (9) into algebraic equations for the coefficients C_i (i = 1, 2, 3, 4):

$$(K + k_3)\,C_{1,3} - k_0 C_{2,4} \mp \lambda C_{4,2} = 0,$$
$$(K - k_3)\,C_{2,4} - k_0 C_{1,3} \pm \lambda C_{3,1} = 0,$$
$$(k_0 \cos\theta \mp \zeta\mathscr{L})\,C_{1,3} \mp \lambda e^{\mp i\theta}\,C_{3,1} = 0,$$

$$(k_0 \cos\theta \mp \zeta\mathscr{L})\,C_{2,4} \pm \lambda e^{\pm i\theta}\,C_{4,2} = 0. \tag{13}$$

These eight equations for the four C_i are compatible under the condition

$$K = \sqrt{k_0^2 + k_3^2 + \lambda^2}, \quad \mathscr{L} = \sqrt{k_0^2 \cos^2\theta + \lambda^2}. \tag{14}$$

The four quantum numbers numbers k_1, k_3, λ and $\zeta = \pm 1$ thus turn out to be independent. Here λ should be determined from system (11). System (13) under condition (14) has the solution

$$C_1 = \frac{1}{2}\sqrt{\left(1 + \zeta\frac{k_0\cos\theta}{\mathscr{L}}\right)\left(1 - \frac{k_3}{K}\right)}\; e^{-i\frac{\theta - \zeta\eta}{2}},$$

$$C_2 = \frac{\zeta}{2}\sqrt{\left(1 + \zeta\frac{k_0\cos\theta}{\mathscr{L}}\right)\left(1 + \frac{k_3}{K}\right)}\; e^{i\frac{\theta - \zeta\eta}{2}},$$

$$C_3 = -\frac{\zeta}{2}\sqrt{\left(1 - \zeta\frac{k_0\cos\theta}{\mathscr{L}}\right)\left(1 - \frac{k_3}{K}\right)}\; e^{i\frac{\theta + \zeta\eta}{2}},$$

$$C_4 = \frac{1}{2}\sqrt{\left(1 - \zeta\frac{k_0\cos\theta}{\mathscr{L}}\right)\left(1 + \frac{k_3}{K}\right)}\; e^{-i\frac{\theta + \zeta\eta}{2}}, \tag{15}$$

where $\sin\eta = k_0 \sin\theta / \sqrt{k_0^2 + \lambda^2}$, and

$$\sum_{i=1}^{4} |c_i|^2 = 1.$$

To determine the explicit dependence on the wave function on the spin (i.e., to separate the solutions on the basis of polarization space), we do not need to know the specific forms of the functions $\Phi\left(\frac{y}{y_0}\right)$ and $\varphi(\alpha t)$; we only need to solve systems (11) and (12).

It follows immediately from system (11) that f and g have the following properties:

$$\frac{d}{dy} g^* f = \lambda (ff^* - gg^*), \text{ i.e., } \int f^* f \, dy = \int g^* g \, dy. \tag{16}$$

Both functions can thus be normalized by the condition

$$\int f^* f \, dy = \int g^* g \, dy = 1. \tag{17}$$

The normalization of the total function is described by

$$\int |\Psi(\vec{r}, t)|^2 d^3x = N, \tag{18}$$

where

$$N = |\chi_1|^2 + |\chi_2|^2 - \frac{k_3}{K}(|\chi_1|^2 - |\chi_2|^2). \tag{19}$$

Differentiating N with respect to the time and using Eqs. (12) we can show that normalization (18) does not depend on the time, for we have $dN/dt = 0$. We can therefore set $N = 1$, thereby normalizing the functions χ_1 and χ_2.

We have thus reduced the solution of the Dirac equation to the solution of a system of first-order differential equations, (11) and (12), and to proceed we need to specify Φ and φ.

Squaring systems (11) and (12), we find second-order differential equations for each of the functions f, g, χ_1 and χ_2:

$$\left\{\frac{d^2}{dy^2} - (\gamma y_0 \Phi - k_1)^2 \mp \gamma \Phi' + \lambda^2\right\} \begin{pmatrix} f \\ g \end{pmatrix} = 0,$$

$$\left\{\frac{1}{c^2}\frac{d^2}{dt^2} + \left(k_3 - \frac{eE_0}{\alpha\hbar}\varphi\right)^2 + \lambda^2 + k_0^2 \pm \frac{ieE_0}{c\hbar}\varphi'\right\} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0. \tag{20}$$

Here the upper signs refer to f and χ_1, while the lower signs refer to g and χ_2. These equations differ from those for a scalar particle (the Klein-Gordon equations) only in the presence of the derivatives Φ' and φ'.

A REMARK ABOUT THE CLASSICAL LORENTZ EQUATIONS

For fields of the form (1) and (2), the classical Lorentz equations can always be solved in quadratures. The Lorentz equations can be written

$$\frac{d^2\vec{r}}{d\tau^2} = -\frac{e}{mc}\left\{\vec{E}\frac{dct}{d\tau} + \left[\frac{d\vec{r}}{d\tau}\vec{H}\right]\right\}, \quad \frac{d^2ct}{d\tau^2} = -\frac{e}{mc}\left(\frac{d\vec{r}}{d\tau}\vec{E}\right), \tag{21}$$

where τ is the proper time. Using (2), we can write the first four integrals of motion:

$$\frac{dx}{d\tau} = c\left(\frac{p_1}{mc} - \frac{eH_0 y_0}{mc^2}\Phi\right), \quad \frac{dz}{d\tau} = c\left(\frac{p_3}{mc} - \frac{eE_0}{mc\alpha}\varphi\right),$$

$$\left(\frac{dy}{d\tau}\right)^2 = c^2\left\{\left(\frac{p_\perp}{mc}\right)^2 - \left(\frac{p_1}{mc} - \frac{eHy_0}{mc^2}\Phi\right)^2\right\},$$

$$\frac{dt}{d\tau} = \left\{1 + \left(\frac{p_\perp}{mc}\right)^2 + \left(\frac{p_3}{mc} - \frac{eE_0}{mc\alpha}\varphi\right)^2\right\}^{1/2}, \tag{22}$$

where p_1, p_3, and $p_\perp$ are constants (the analogs of the quantum numbers k_1, k_3 and λ). Using (22), we can write a solution of the problem in quadratures:

$$x - x_0 = \int \frac{\left[\frac{p_1}{mc} - \frac{eH_0 y_0}{mc^2}\Phi\left(\frac{y}{y_0}\right)\right] dy}{\sqrt{\left(\frac{p_\perp}{mc}\right)^2 - \left[\frac{p_1}{mc} - \frac{eHy_0}{mc^2}\Phi\left(\frac{y}{y_0}\right)\right]^2}}, \tag{23}$$

$$z - z_0 = c \int \frac{\left[\frac{p_3}{mc} - \frac{eE_0}{mc\alpha}\varphi(\alpha t)\right] dt}{\sqrt{1 + \left(\frac{p_\perp}{mc}\right)^2 + \left[\frac{p_3}{mc} - \frac{eE_0}{mc\alpha}\varphi(\alpha t)\right]^2}},$$

$$\int \left\{ \left(\frac{p_\perp}{mc}\right)^2 - \left[\frac{p_1}{mc} - \frac{eH_0 y_0}{mc^2}\Phi\left(\frac{y}{y_0}\right)\right]^2 \right\}^{-1/2} dy =$$

$$= c \int \left\{ 1 + \frac{p_\perp}{mc} + \frac{p_3}{mc} - \frac{eE_0}{mc\alpha}\varphi(\alpha t) \right\}^{-1/2} dt. \tag{23}$$

Here y and z are found as functions of t, and x is found as a function of y, which in turn depends on t.

SOME PARTICULAR FIELDS

An important case is that of a uniform magnetic field, with $\Phi = \frac{y}{y_0}$. The solution of system (11) is [2]

$$f = \sqrt[4]{\gamma}\, U_{n-1}(\xi);\ g = \sqrt[4]{\gamma}\, U_n(\xi);\ \xi = \sqrt{\gamma}\left(y - \frac{k_1}{\gamma}\right),$$

$$\lambda = \sqrt{2n\gamma}, \quad n = 0, 1, 2 \ldots, \tag{24}$$

where $U_n(\xi)$ are the Hermite functions, related to the Hermite polynomials by

$$U_n(\xi) = (2^n n! \sqrt{\pi})^{-1/2} e^{-\xi^2/2} H_n(\xi). \tag{25}$$

(See [4] for solutions of the system (11) for certain cases of nonuniform fields.)

Equations (12) can also be solved exactly, e.g., in the case $\varphi(\alpha t) = \operatorname{th}\alpha t$. This choice of potential corresponds to an electric field $E_z = E_0(\operatorname{ch}\alpha t)^{-2}$, which vanishes as $t \to \pm\infty$. In this case the quantity α^{-1} characterizes the time for which the field is effective. Then the solution of Eqs. (12) can be expressed in terms of the Gaussian hypergeometric function:

$$\chi_1 = Q(q - p - 2\delta)\left(\frac{\alpha q + ck_3 - \alpha\delta}{cK - ck_3}\right)^{1/2} \xi_1^{-ip/2} \times$$

$$\times (1 - \xi_1)^{iq/2} F\left[1 + \frac{i}{2}(q - p + 2\delta);\right.$$

$$\left.\frac{i}{2}(q - p - 2\delta);\ 1 - ip;\ \xi_1\right].$$

$$\chi_2 = -Q(q - p - 2\delta)\left(\frac{\alpha q - ck_3 + \alpha\delta}{cK + ck_3}\right)^{1/2} \xi_1^{-ip/2}(1 - \xi_1)^{iq/2} \times$$

$$\times F\left[\frac{i}{2}(q - p + 2\delta);\ (1 + \frac{i}{2}q - p - 2\delta);\ 1 - ip;\ \xi_1\right]. \tag{26}$$

Here we have

$$q = \frac{c}{\alpha}\sqrt{\lambda^2 + k_0^2 + \left(k_3 - \frac{\alpha}{c}\delta\right)^2},$$

$$p = \frac{c}{\alpha}\sqrt{\lambda^2 + k_0^2 + \left(k_3 + \frac{\alpha}{c}\delta\right)^2},$$

$$\delta = \frac{eE_0 c}{\hbar\alpha^2} \text{ and } \xi_1 = \frac{1}{2}(1 + \operatorname{th}\alpha t). \tag{27}$$

The factor Q is arbitrary and must be found from the normalization condition. We note that functions (27) convert into the steady-state solutions as $t \to \pm\infty$, i.e., when there is no electric field. As $t \to -\infty$ we find from (27) that

$$\chi_1 \sim \chi_2 \sim \exp\left\{-ict\sqrt{\lambda^2 + k_0^2 + \left(k_3 + \frac{\alpha}{c}\delta\right)^2}\right\}, \tag{28}$$

while as $t \to \infty$ we find that

$$\chi_1 \sim \chi_2 \sim \exp\left\{-ict\sqrt{\lambda^2 + k_0^2 + \left(k_3 - \frac{\alpha}{c}\delta\right)^2}\right\}. \tag{29}$$

The energy shift during a time change from $-\infty$ to ∞ is the same as that predicted by classical theory (see (23)).

With $E_0 = 0$, these functions obviously converge into the stationary-state wave functions:

$$\chi_1 \sim \chi_2 \sim e^{-icKt}.$$

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