FREQUENCY CORRELATIONS IN THE FLUCTUATIONS OF SPHERICAL WAVES PROPAGATING IN A TURBULENT MEDIUM

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The frequency correlations are calculated for the amplitude and phase fluctuations of two spherical waves of different frequencies propagating in an isotropic turbulent medium. The structure function of the refractive index of such a medium obeys a Kolmogorov-Obukhov two-thirds law. The analysis is carried out from the wave point of view in the first approximation of the smooth-perturbation method.

The turbulence of the atmosphere, ionosphere, and oceans which causes fluctuations in the refractive index leads to fluctuations in the properties of waves propagating in these mediums. Many studies have been devoted to the statistical characteristics of these fluctuations. In addition to the spatial correlation of the fluctuations in the properties of a wave, a single frequency, the frequency and frequency-spatial correlation functions have been studied. By studying these latter correlation functions we have determine the frequency band which can propagate through a statistically inhomogeneous medium without distortion of signal shape: this determination is required in order to evaluate the appropriate frequency bands in communication systems employing frequency-diversity or frequency-and-spatial-diversity. The use of highly directional antennas also requires knowledge of the spatial and frequency correlations in the field fluctuations in the aperture plane of the antenna for determining the appropriate frequency band.

In previous calculations [1-6] of the frequency-spatial correlation functions for the fluctuations in wave properties in the atmosphere and the ionosphere, a statically homogeneous and isotropic medium having a Gaussian function for the fluctuations in the refractive index has been assumed. The refractive index of the actual turbulent atmosphere is described satisfactorily by means of a local homogeneous field having constants for slowly varying average properties. The structure function for the refractive index of such a medium obeys the Kolmogorov-Obukhov two-thirds law. Experimental data implied that sea water is also best described by a homogeneous-turbulence model. The frequency-correlation coefficient was calculated in [7] for the amplitude fluctuations of waves propagating in such a medium. The transverse spectra and structure functions were calculated in [8], for the amplitude and phase fluctuations and for the correlation in these fluctuations for two plane waves of different frequencies; the effects of the receiver-aperture size on the frequency fluctuations in the field intensity were also taken into account.

The theoretical studies in [1-8] dealt with the propagation of two plane waves of different frequency, but experimentally we are forced to deal with sources which radiate spherical or approximately spherical waves. We know of no theoretical study of the spatial and frequency correlations in the fluctuations of spherical-wave properties.

Below we will use the wave approach to analyze the correlation between the amplitude and phase fluctuations of two spherical waves of different frequencies. We will solve the problem in the first approximation of the smooth-perturbation method. We assume turbulent medium to be homogeneous and isotropic, having a structure function for the refractive index which satisfies the two-thirds law.

We assume a source emitting two spherical monochromatic waves with wave numbers $k_1=k(1-\delta)$ and $k_2=k(1+\delta)$ at the origin of a Cartesian coordinate system. The quantity $\delta=\frac{k_2-k_1}{2k}$ characterizes the frequency distance between the waves. The refractive index of the medium is $n(x, y, z)=1+\mu(x, y, z)$, where $\mu(x, y, z)$ are airy small random deviations from the average value, which is assumed equal to unity ($|\mu|\ll 1$).

We assume that the inhomogeneities are distributed in a quasi static manner, i.e., that the properties of the medium change so slowly that the time dependence of can be neglected; we also assume that the inhomogeneities themselves are airy large-scale, i.e., that $k_{1,2}l_0\gg 1$, where l_0 is the internal scale of the turbulence.

The field of each wave in the medium is described by a wave equation

$$\Delta u+k^2(1+\mu)^2u=0. \tag{1}$$

For brevity, we will not display here the familiar general solution of this equation found by the smooth-perturbation method; instead we will use an expression found for the complex phase in the first approximation of this method* [9]

*The complex phase is related to the magnitude of the field itself by $u=\exp\{\Phi_0+\Phi\}$, where Φ_0 is the complex phase of the zeroth approximation field.

$$\Phi(x, y, z) = \frac{k^2}{4\pi} \iiint_V \frac{\exp[ik\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}]}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \times$$
$$\times \frac{u_0(x', y', z')}{u_0(x, y, z)} \mu(x', y', z')\, dx'\, dy'\, dz', \tag{2}$$

where V is the region containing the inhomogeneities, and u_0, which corresponds to the unperturbed wave, satisfies the wave equation

$$\Delta u_0 + k^2 u_0 = 0. \tag{3}$$

Since the unperturbed wave is spherical and is emitted from the origin we have

$$\frac{u_0(x', y', z')}{u_0(x, y, z)} = \frac{\sqrt{x^2+y^2+z^2}}{\sqrt{x'^2+y'^2+z'^2}} \exp[ik\sqrt{x'^2+y'^2+z'^2} -$$
$$- ik\sqrt{x^2+y^2+z^2}].$$

Since we are dealing with large-scale inhomogeneities, the field at the observation point will be determined by wave scattering in the medium within a narrow ellipsoid having foci at the source and receiver. In the region important for the integration, we can thus set $|x-x'| \gg |y-y'|,\ |z-z'|$

$$\exp[ik\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}] \simeq \exp\left\{ik\left[(x-x') + \right.\right.$$
$$\left.\left. + \frac{1}{2}\frac{(y-y')^2+(z-z')^2}{x-x'}\right]\right\},$$
$$\exp[ik\sqrt{x'^2+y'^2+z'^2} - ik\sqrt{x^2+y^2+z^2}] \simeq$$
$$\simeq \exp\left\{ik\left[x' + \frac{y'^2+z'^2}{2x'} - x - \frac{y^2+z^2}{2x}\right]\right\},$$
$$\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \simeq \frac{1}{x-x'},$$
$$\frac{\sqrt{x^2+y^2+z^2}}{\sqrt{x'^2+y'^2+z'^2}} \simeq \frac{x}{x'}$$

and extend the x' from 0 to x and extend the y' and z' integrations to infinite limits. Then the complex phase becomes

$$\Phi(x, y, z) = \frac{k^2 x}{4\pi} \int_0^x \frac{dx'}{x'(x-x')} \iint_{-\infty}^{+\infty} \exp\left\{\frac{ik}{2}\left[\frac{(y-y')^2+(z-z')^2}{x-x'} + \right.\right.$$
$$\left.\left. + \frac{y'^2+z'^2}{x'} - \frac{y^2+z^2}{x}\right]\right\} \mu(x', y', z')\, dy'\, dz'. \tag{4}$$

To determine the frequency structure and correlation functions for the amplitude and phase fluctuations in the plane perpendicular to the propagation directions, we will calculate $\langle \Phi_1(x, y_1, z_1) \Phi_2^*(x, y_2, z_2) \rangle$ and $\langle \Phi_1(x, y_1, z_1) \Phi_2(x, y_2, z_2) \rangle$. Here the asterisk denotes the complex conjugate, and the angle denote an averaging over the ensemble of states. The right-hand subscripts refer to different frequencies:

$$\left.\begin{matrix} \langle \Phi_1(x, y_1, z_1) \Phi_2^*(x, y_2, z_2) \rangle \\ \langle \Phi_1(x, y_1, z_1) \Phi_2(x, y_2, z_2) \rangle \end{matrix}\right\} = \frac{k^4 x^2}{16\pi^2}(1-\delta^2)^2 \int_0^x \int_0^x \frac{dx'\, dx''}{x'(x-x')\, x''(x-x'')}.$$
$$\iiiint_{-\infty}^{+\infty} \exp\left\{\frac{ik}{2}\left[\frac{(y_1-y')^2+(z_1-z')^2}{x-x'} + \frac{z'^2+z'^2}{x'} - \frac{y_1^2+z_1^2}{x}\right](1-\delta) \mp \right.$$
$$\left. \mp \left[\frac{(y_2-y'')^2+(z_2-z'')^2}{x-x''} + \frac{y''^2+z''^2}{x''} - \frac{y_2^2+z_2^2}{x}\right]\frac{ik}{2}(1+\delta)\right\} \times$$
$$\times \langle \mu(x', y', z') \mu(x'', y'', z'') \rangle\, dy'\, dy''\, dz'\, dz''. \tag{5}$$

Introducing the notation

$$B_n(x'-x'', y'-y'', z'-z'') = \langle \mu(x', y', z')\mu(x'', y'', z'')\rangle; \tag{6}$$

introducing the new variables $\eta = y'-y''$, $\zeta = z'-z''$, $u = \frac{y'+y''}{2}$ and $v = \frac{z'+z''}{2}$; and using the familiar relation

$$\int_{-\infty}^{+\infty} \exp(-ax^2 \pm bx)\, dx = \sqrt{\frac{\pi}{a}} \exp\frac{b^2}{4a}; \tag{6'}$$

we find, after straightforward but lengthy calculations,

$$\langle \Phi_1(x, y_1, z_1)\Phi_2^*(x, y_2, z_2)\rangle =$$

$$= -\frac{ik^3x}{8\pi}\int_0^x\int_0^x \frac{(1-\delta^2)^2\,dx'\,dx''}{(x'-x'')(x-x'-x'') + [x(x'+x'') - (x'^2+x''^2)]\,\delta} \times$$

$$\times \iint_{-\infty}^{+\infty} \exp\left\{\frac{ikx}{2}\,\frac{(1-\delta^2)\left[\left(\eta - \frac{y_1x'}{x} + \frac{y_2x''}{x}\right)^2 + \left(\zeta - \frac{z_1x'}{x} + \frac{z_2x''}{x}\right)^2\right]}{(x'-x'')(x-x'-x'') + [x(x'+x'') - (x'^2+x''^2)]\,\delta}\right\} \times$$

$$\times B_n(x'-x'', \eta, \zeta)\, d\eta\, d\zeta.$$

$$\langle \Phi_1(x, y_1, z_1)\Phi_2(x, y_2, z_2)\rangle =$$

$$= \frac{ik^3x}{8\pi}\int_0^x\int_0^x \frac{(1-\delta^2)^2\,dx'\,dx''}{x(x'+x'') - (x'^2+x''^2) + (x'-x'')(x-x'-x'')\,\delta} \times$$

$$\times \iint_{-\infty}^{+\infty} \exp\left\{\frac{ikx}{2}\,\frac{(1-\delta^2)\left[\left(\eta - \frac{y_1x'}{x} + \frac{y_2x''}{x}\right)^2 + \left(\zeta - \frac{z_1x'}{x} + \frac{z_2x''}{x}\right)^2\right]}{x(x'+x'') - (x'^2+x''^2) + (x'-x'')(x-x'-x'')\,\delta}\right\} \times$$

$$\times B_n(x'-x'', \eta, \zeta)\, d\eta\, d\zeta.$$

Using the two-dimensional spectral decomposition of $B_n(x'-x'', \eta, \zeta)$,

$$B_n(x'-x'', \eta, \zeta) = \iint_{-\infty}^{+\infty} \exp[i(\varkappa_2\eta + \varkappa_3\zeta)]\, F_n(\varkappa_2, \varkappa_3, x'-x'')\, d\varkappa_2\, d\varkappa_3, \tag{7}$$

where $\varkappa_2$ and $\varkappa_3$ are the spatial wave numbers, we find (6') that

$$\langle \Phi_1(x, y_1, z_1)\Phi_2^*(x, y_2, z_2)\rangle = \frac{k^2}{4}(1-\delta^2)\int_0^x\int_0^x dx'\,dx'' \iint_{-\infty}^{+\infty} F_n(\varkappa_2, \varkappa_3, x'-x)\times$$

$$\times \exp\left\{(\varkappa_2^2+\varkappa_3^2)\,\frac{(x'-x'')(x-x'-x'') + [x(x'+x'') - (x'^2+x''^2)]\,\delta}{2ikx(1-\delta^2)} + \right.$$

$$\left. + i\left(\frac{y_1x'}{x} - \frac{y_2x''}{x}\right)\varkappa_2 + i\left(\frac{z_1x'}{x} - \frac{z_2x''}{x}\right)\varkappa_3\right\} d\varkappa_2\, d\varkappa_3.$$

$$\langle \Phi_1(x, y_1, z_1)\Phi_2(x, y_2, z_2)\rangle = -\frac{k^2}{4}(1-\delta^2)\iint_0^{x\,x} dx'dx'' \iint_{-\infty}^{+\infty} F_n(\varkappa_2, \varkappa_3, x'-x)\times$$

$$\times \exp\left\{(\varkappa_2^2+\varkappa_3^2)\,\frac{x(x'+x'') - (x'^2+x''^2) + (x'-x'')(x-x'-x'')\,\delta}{2ikx(1-\delta^2)} + \right.$$

$$\left. + i\left(\frac{y_1x'}{x} - \frac{y_2x''}{x}\right)\varkappa_2 + i\left(\frac{z_1x'}{x} - \frac{z_2x''}{x}\right)\varkappa_3\right\} d\varkappa_2\, d\varkappa_3.$$

Since $F_n(\varkappa_2, \varkappa_3, x'-x'')$ is significantly different from zero only in the region $\varkappa(x'-x'') \leqslant 1$ (where $\varkappa = \sqrt{\varkappa_2^2+\varkappa_3^2}$), we have

$$\frac{\varkappa^2(x'-x'')(x-x'-x'')}{2kx}\,\frac{1}{1-\delta^2} \leqslant \frac{\varkappa(x-x'-x'')}{2kx}\,\frac{1}{1-\delta^2} \sim$$

$$\sim \frac{\varkappa}{k} \leqslant \frac{\lambda}{l_0} \ll 1$$

because the wave length λ is small with the internal scale l_0 of the turbulence.

Similarly, we have

$$\frac{\varkappa^2 (x'-x'')(x-x'-x'')}{2kx} \frac{\delta}{1-\delta^2} \ll 1,$$

and we can assume that

$$\exp\left\{(\varkappa_2^2+\varkappa_3^2)\frac{(x'-x'')(x-x'-x'')}{2kx}\, p\right\} \approx 1,$$

where

$$p = \frac{1}{1-\delta^2}, \quad \frac{\delta}{1-\delta^2}.$$

Then we have

$$\langle \Phi_1(x, y_1, z_1)\Phi_2^*(x, y_2, z_2)\rangle = \frac{k^2}{4}(1-\delta^2)\int_0^x\int_0^x dx'\,dx'' \iint_{-\infty}^{+\infty} F_n(\varkappa_2, \varkappa_3, x'-x'')\times$$
$$\times \exp\left\{(\varkappa_2^2+\varkappa_3^2)\frac{x(x'+x'')-(x'^2+x''^2)}{2ikx}\frac{\delta}{1-\delta^2}+\right.$$
$$\left.+\,i\left(\frac{y_1 x'}{x}-\frac{y_2 x''}{x}\right)\varkappa_2 + i\left(\frac{z_1 x'}{x}-\frac{z_2 x''}{x}\right)\varkappa_3\right\} d\varkappa_2\, d\varkappa_3. \quad (8)$$

$$\langle \Phi_1(x, y_1, z_1)\Phi_2(x, y_2, z_2)\rangle = -\frac{k^2}{4}(1-\delta^2)\int_0^x\int_0^x dx'\,dx'',$$

$$\iint_{-\infty}^{+\infty} F_n(\varkappa_2, \varkappa_3, x'-x'')\exp\left\{(\varkappa_2^2+\varkappa_3^2)\frac{x(x'+x'')-(x'^2+x''^2)}{2ikx}\frac{1}{1-\delta^2}+\right.$$
$$\left.+\,i\left(\frac{y_1 x'}{x}-\frac{y_2 x''}{x}\right)\varkappa_2 + i\left(\frac{z_1 x'}{x}-\frac{z_2 x''}{x}\right)\varkappa_3\right\} d\varkappa_2\, d\varkappa_3. \quad (9)$$

Introducing the new variables

$$\xi = x'-x'', \quad v = \frac{x'+x''}{2},$$

we convert the first terms in the exponential functions in Eqs. (8) and (9) to

$$\frac{\varkappa^2[x(x'+x'')-(x'^2+x''^2)]}{2ikx}\, p = \frac{\varkappa^2}{2ikx}\, p\left[2vx-2v^2-\frac{\xi^2}{2}\right],$$

where

$$p = \frac{\delta}{1-\delta^2}, \quad \frac{1}{1-\delta^2},$$
$$\varkappa^2 = \varkappa_2^2+\varkappa_3^2.$$

As we already mentioned, however, we have $\varkappa\xi \ll 1$ in the region important for the integration, so the quantity $\xi^2/2$ in this last expression is negligible. Since we have $\xi \leqslant \sqrt{\lambda x} \ll x$, we can assume

$$i\left(\frac{y_1 x'}{x}-\frac{y_2 x''}{x}\right)\varkappa_2 \simeq i\,\frac{(y_1-y_2)v]}{x}\varkappa_2,$$
$$i\left(\frac{z_1 x'}{x}-\frac{z_2 x''}{x}\right)\varkappa_3 \approx i\,\frac{(z_1-z_2)v}{x}\varkappa_3.$$

Extending the ξ integration to infinite limits by virtue of the rapid decay of $F_n(\varkappa_2, \varkappa_3, \xi)$, and using

$$\int_0^\infty F_n(\varkappa_2, \varkappa_3, \xi)\, d\xi = \pi\Phi_n(0, \varkappa_2, \varkappa_3) \quad (10)$$

where $\Phi_n(0, \varkappa_2, \varkappa_3)$ is the three-dimensional spectral density of the refractive index, we find

$$\langle \Phi_1(x, y_1, z_1)\Phi_2^*(x, y_2, z_2)\rangle = \frac{\pi k^2}{2}(1-\delta^2)\int_0^x dv \iint_{-\infty}^{+\infty}\Phi_n(0, \varkappa_2, \varkappa_3)\times$$
$$\times\exp\left\{(\varkappa_2^2+\varkappa_3^2)\frac{v(x-v)}{ikx}\frac{\delta}{1-\delta^2}+i\,\frac{(y_1-y_2)v]}{x}\varkappa_2+\right.$$
$$\left.+\,i\,\frac{(z_1-z_2)v}{x}\varkappa_3\right\} d\varkappa_2\, d\varkappa_3, \quad (11)$$

$$\langle \Phi_1(x, y_1, z_1)\Phi_2(x, y_2, z_2)\rangle = -\frac{\pi k^2}{2}(1-\delta^2)\int_0^x dv \iint_{-\infty}^{+\infty} \Phi_n(0, \varkappa_2, \varkappa_3) \times$$
$$\times \exp\left\{(\varkappa_2^2+\varkappa_3^2)\frac{v(x-v)}{ikx}\,\frac{1}{1-\delta^2} + i\frac{(y_1-y_2)v}{x}\varkappa_2 + \right.$$
$$\left. + i\frac{(z_1-z_2)v}{x}\varkappa_3\right\} d\varkappa_2\, d\varkappa_3. \qquad (12)$$

Since the refractive index is isotropic in this case, we have

$$\Phi_n(0, \varkappa_2, \varkappa_3) = \Phi_n\left(0, \sqrt{\varkappa_2^2+\varkappa_3^2}\right),$$

Introducing the new integration variables $\varkappa_2 = \varkappa\cos\varphi$ and $\varkappa_3 = \varkappa\sin\varphi$ and writing $y_1 - y_2 = \rho\cos\alpha$, we find

$$z_1 - z_2 = \rho\sin\alpha \quad \text{and} \quad \rho = \sqrt{(y_1-y_2)^2 + (z_1-z_2)^2},$$

Using the familiar expression

$$\int_{-\pi}^{+\pi} \cos[\varkappa\rho\cos(\varphi-\alpha)]\, d\varphi = 2\pi I_0(\varkappa\rho),$$

where I_0 is the zero-index Bessel function of the first kind, we carry out the φ integration; then Eqs. (11) and (12) become

$$\langle \Phi_1(x, y_1, z_1)\Phi_2^*(x, y_2, z_2)\rangle =$$
$$= 2\pi^2 k^2(1-\delta^2)\int_0^x dv \int_0^\infty \varkappa\Phi_n(\varkappa)\exp\left[\varkappa^2\frac{v(x-v)}{ikx}\,\frac{\delta}{1-\delta^2}\right].$$
$$I_0\left(\varkappa\rho\frac{v}{x}\right)d\varkappa; \qquad (13)$$

$$\langle \Phi_1(x, y_1, z_1)\Phi_2(x, y_2, z_2)\rangle =$$
$$= -2\pi^2 k^2(1-\delta^2)\int_0^x dv \int_0^\infty \varkappa\Phi_n(\varkappa)\exp\left\{\varkappa^2\frac{v(x-v)}{ikx}\,\frac{1}{1-\delta^2}\right\}.$$
$$I_0\left(\varkappa\rho\frac{v}{x}\right)d\varkappa. \qquad (14)$$

The joint structure functions for the amplitude and phase fluctuations and for the correlation between these fluctuations for two plane waves of different frequencies are given by

$$D_{\chi_1\chi_2}(r_1, r_2) = \langle[\chi_1(r_1) - \chi_1(r_2)][\chi_2(r_1) - \chi_2(r_2)]\rangle =$$
$$= \frac{1}{2}\operatorname{Re}[D_1(r_1, r_2) + D_2(r_1, r_2)],$$
$$D_{S_1S_2}(r_1, r_2) = \langle[S_1(r_1) - S_1(r_2)][S_2(r_1) - S_2(r_2)]\rangle =$$
$$= \frac{1}{2}\operatorname{Re}[D_1(r_1, r_2) - D_2(r_1, r_2)],$$
$$D_{\chi_1S_2}(r_1, r_2) = \langle[\chi_1(r_1) - \chi_1(r_2)][S_2(r_1) - S_2(r_2)]\rangle =$$
$$= \frac{1}{2}Im[D_2(r_1, r_2) - D_1(r_1, r_2)],$$
$$D_{\chi_2S_1}(r_1, r_2) = \langle[\chi_2(r_1) - \chi_2(r_2)][S_1(r_1) - S_1(r_2)]\rangle =$$
$$= \frac{1}{2}Im[D_2(r_1, r_2) + D_1(r_1, r_2)], \qquad (15)$$

where the structure function of the complex phases are

$$D_1(r_1, r_2) = \langle[\Phi_1(r_1) - \Phi_1(r_2)][\Phi_2^*(r_1) - \Phi_2^*(r_2)]\rangle,$$
$$D_2(r_1, r_2) = \langle[\Phi_1(r_1) - \Phi_1(r_2)][\Phi_2(r_1) - \Phi_2(r_2)]\rangle \qquad (16)$$

and r_1 and r_2 are points having coordinates (x, y_1, z_1) and (x, y_2, z_2).

For this isotropic-turbulence case, we write the three-dimensional spectral density of the refractive-index fluctuation as

$$\Phi_n(\varkappa) = 0{,}033\, C_n^2\, \varkappa^{-11/3}, \tag{17}$$

where C_n is structure constant of the refractive index. Substituting Eqs. (13), (14), and (17) into (16) we find

$$D_1(r_1, r_2) = 4\pi^2\, 0{,}033\, C_n^2 k^2 (1-\delta^2) \int_0^x dv \int_0^\infty \varkappa^{-8/3} \exp\left[\varkappa^2 \frac{v(x-v)}{ikx} \frac{\delta}{1-\delta^2}\right] \times \left[1 - I_0\left(\varkappa\rho \frac{v}{x}\right)\right] d\varkappa, \tag{18}$$

$$D_2(r_1, r_2) = -4\pi^2\, 0{,}033\, C_n^2 k^2 (1-\delta^2) \int_0^x dv \int_0^\infty \varkappa^{-8/3} \exp\left[\varkappa^2 \frac{v(x-v)}{ikx} \frac{1}{1-\delta^2}\right] \times \left[1 - I_0\left(\varkappa\rho \frac{v}{x}\right)\right] d\varkappa. \tag{19}$$

We use the value of this integral in the form [9

$$\int_0^\infty [1 - I_0(\varkappa\rho)]\, \varkappa^{-q} \exp\left(-\frac{\varkappa^2}{\varkappa_m^2}\right) d\varkappa = -\frac{1}{2}\Gamma\left(-\frac{q-1}{2}\right)\varkappa_m^{-(q-1)}\left[{}_1F_1\left(-\frac{q-1}{2},\ 1,\ -\frac{\varkappa_m^2\rho^2}{4}\right) - 1\right], \tag{20}$$

where Γ is the gamma function, and ${}^1F_1$ is the degenerate hypergeometric function.

The use of $w = v/(x-v)$ in Eqs. (18) and (19) results in

$$D_1(r_1, r_2) = A\delta^{5/6} i^{5/6} \left\{x^{11/6} \int_0^\infty w^{5/6} (1+w)^{-11/3} \times {}_1F_1\left(-\frac{5}{6},\ 1,\ \frac{ik\rho^2}{4x} \frac{1-\delta^2}{\delta} w\right) dw - \int_0^x \left[\frac{v(x-v)}{x}\right]^{5/6} dv\right\}, \tag{21}$$

$$D_2(r_1, r_2) = -Ai^{5/6} \left\{x^{11/6} \int_0^\infty w^{5/6} (1+w)^{-11/3} \times {}_1F_1\left(-\frac{5}{6},\ 1,\ i\frac{k\rho^2}{4x}(1-\delta^2) w\right) dw - \int_0^x \left[\frac{v(x-v)}{x}\right]^{5/6} dv\right\}, \tag{22}$$

where

$$A = -2\pi^2\, 0{,}033\, C_n^2\, \Gamma\left(-\frac{5}{6}\right) k^{7/6} (1-\delta^2)^{1/6}.$$

Using (/10, pp. 964 and 11)

$$\int_0^1 y^{5/6}(1-y)^{5/6}\, dy = \frac{6}{11}\, {}_2F_1\left(\frac{11}{6},\ -\frac{5}{6},\ \frac{17}{6},\ 1\right),$$

$$\int_0^\infty y^{5/6}(1+y)^{-11/3}\, {}_1F_1\left(-\frac{5}{6},\ 1,\ i\frac{k\rho^2}{4x} y\right) dy = \frac{\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{11}{6}\right)}{\Gamma\left(\frac{11}{3}\right)}\, {}_1F_1\left(\frac{11}{6},\ 1,\ -i\frac{k\rho^2}{4x}\right) - \frac{6}{11\,\Gamma\left(\frac{17}{6}\right)}\left[-i\frac{k\rho^2}{4x}\right]^{11/6} {}_2F_2\left(\frac{11}{3},\ 1;\ \frac{17}{6},\ \frac{17}{6};\ -i\frac{k\rho^2}{4x}\right)$$

and Eqs. (15), we find

$$D_{\substack{\chi_1\chi_2\\S_1S_2}}(\rho) = \frac{1}{2} A x^{11/6} \mathrm{Re}\, i^{5/6} \left\{ \frac{\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{11}{6}\right)}{\Gamma\left(\frac{11}{3}\right)} \left[\delta^{5/6} {}_1F_1\left(\frac{11}{6},\ 1,\ Z\frac{1-\delta^2}{\delta}\right) \mp \right.\right.$$

$$\left. \mp {}_1F_1\left(\frac{11}{6},\ 1,\ Z(1-\delta^2)\right)\right] - \frac{6}{11}\frac{Z^{11/6}}{\Gamma\left(\frac{17}{6}\right)}\left[\delta^{5/6}\left(\frac{1-\delta^2}{\delta}\right)^{11/6} \times \right.$$

$$\times {}_2F_2\left(\frac{11}{3},\ 1;\ \frac{17}{6},\ \frac{17}{6};\ Z\frac{1-\delta^2}{\delta}\right) \mp$$

$$\left. \mp (1-\delta^2)^{11/6} {}_2F_2\left(\frac{11}{3},\ 1;\ \frac{17}{6},\ \frac{17}{6};\ Z(1-\delta^2)\right)\right] +$$

$$\left. + \frac{6}{11}(1 \pm \delta^{5/6})\, {}_2F_1\left(\frac{11}{6},\ -\frac{5}{6},\ \frac{17}{6},\ 1\right)\right\}, \qquad (23)$$

$$D_{\substack{\chi_1S_2\\ \chi_2S_1}}(\rho) = \frac{1}{2} A x^{11/6}\, Im\, i^{5/6} \left\{ \frac{\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{11}{6}\right)}{\Gamma\left(\frac{11}{3}\right)} \times \right.$$

$$\times \left[\mp \delta^{5/6} {}_1F_1\left(\frac{11}{6},\ 1,\ Z\frac{1-\delta^2}{\delta}\right) - \right.$$

$$\left. - {}_1F_1\left(\frac{11}{6},\ 1,\ Z(1-\delta^2)\right)\right] + \frac{6}{11}\frac{Z^{11/6}}{\Gamma\left(\frac{17}{6}\right)}\left[\pm \delta^{5/6}\left(\frac{1-\delta^2}{\delta}\right)^{11/6} \times \right.$$

$$\times {}_2F_2\left(\frac{11}{3},\ 1;\ \frac{17}{6},\ \frac{17}{6};\ Z\frac{1-\delta^2}{\delta}\right) +$$

$$\left. + (1-\delta^2)^{11/6} {}_2F_2\left(\frac{11}{3},\ 1;\ \frac{17}{6},\ \frac{17}{6};\ Z(1-\delta^2)\right)\right] +$$

$$\left. + \frac{6}{11}(1 \pm \delta^{5/6})\, {}_2F_1\left(\frac{11}{6},\ -\frac{5}{6},\ \frac{17}{6},\ 1\right)\right\}, \qquad (24)$$

where

$$Z = -i\frac{k\rho^2}{4x}.$$

Let us take a more detailed look at the frequency correlation for the amplitude fluctuations. As $\rho \to \infty$ we have

$$D_{\chi_1\chi_2}(\infty) = \frac{1}{2} A x^{11/6} (1-\delta^{5/6})\, {}_2F_1\left(\frac{11}{6},\ -\frac{5}{6},\ \frac{17}{6},\ 1\right) \qquad (25)$$

and the frequency correlation function for the amplitude fluctuations become

$$B_{\chi_1\chi_2}(\rho) = \frac{1}{2}\left[D_{\chi_1\chi_2}(\infty) - D_{\chi_1\chi_2}(\rho)\right] =$$

$$= P k^{7/6} x^{11/6} (1-\delta^2) \left\{ \left(\frac{1}{1-\delta^2}\right)^{5/6} \left[\mathrm{Re}\, {}_1F_1\left(\frac{11}{6},\ 1,\ Z(1-\delta^2)\right) - \right.\right.$$

$$- \mathrm{tg}\frac{5\pi}{12}\, Im\, {}_1F_1\left(\frac{11}{6},\ 1,\ Z(1-\delta^2)\right) - Q\left[\frac{k\rho^2}{4x}(1-\delta^2)\right]^{11/6} \times$$

$$\left. \times Im\, {}_2F_2\left(\frac{11}{3},\ 1;\ \frac{17}{6},\ \frac{17}{6};\ Z(1-\delta^2)\right)\right] -$$

$$- \left(\frac{\delta}{1-\delta^2}\right)^{5/6} \left[\mathrm{Re}\, {}_1F_1\left(\frac{11}{6},\ 1,\ Z\frac{1-\delta^2}{\delta}\right) - \right.$$

$$- \mathrm{tg}\frac{5\pi}{12}\, Im\, {}_1F_1\left(\frac{11}{6},\ 1,\ Z\frac{1-\delta^2}{\delta}\right) -$$

$$\left.\left. - Q\left[\frac{k\rho^2}{4x}\,\frac{1-\delta^2}{\delta}\right]^{11/6} Im\, {}_2F_2\left(\frac{11}{3},\ 1;\ \frac{17}{6},\ \frac{17}{6};\ Z\frac{1-\delta^2}{\delta}\right)\right]\right\}, \qquad (26)$$

where

$$P=\frac{3}{5}\,0{,}033\,C_n^2\,\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{11}{16}\right)}{\Gamma\left(\frac{11}{3}\right)}\cos\frac{5\pi}{12},$$

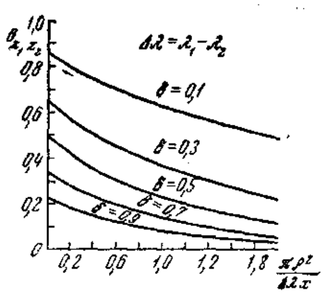
$$Q=\frac{6\Gamma\left(\frac{11}{3}\right)}{11\Gamma\left(\frac{17}{6}\right)\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{11}{6}\right)\cos\frac{5\pi}{12}}.$$

The normalized frequency-correlation coefficient for the amplitude fluctuations is

$$\begin{aligned}b_{\chi_1\chi_2}(\rho)=\frac{B_{\chi_1\chi_2}(\rho)}{\sqrt{B_{\chi_1}(\rho)B_{\chi_2}(\rho)}}=\frac{1}{(1-\delta^2)^{5/12}}\Big\{&\mathrm{Re}\,{}_1F_1\left(\frac{11}{6},\,1,\,Z(1-\delta^2)\right)-\\&-\mathrm{tg}\,\frac{5\pi}{12}\,Im\,{}_1F_1\left(\frac{11}{6},\,1,\,Z(1-\delta^2)\right)-\\&-Q\left[\frac{k\rho^2}{4x}(1-\delta^2)\right]^{11/6}Im\,{}_2F_2\left(\frac{11}{6},\,1;\,\frac{17}{6},\,\frac{17}{6};\,Z(1-\delta^2)\right)-\\&-\delta^{5/6}\,\mathrm{Re}\,{}_1F_1\left(\frac{11}{6},\,1,\,Z\frac{1-\delta^2}{\delta}\right)+\delta^{5/6}\,\mathrm{tg}\,\frac{5\pi}{12}\,Im\,{}_1F_1\left(\frac{11}{6},\,1,\,Z\frac{1-\delta^2}{\delta}\right)+\\&+\delta^{5/6}\,Q\left[\frac{k\rho^2}{4x}\,\frac{1-\delta^2}{\delta}\right]^{11/6}Im\,{}_2F_2\left(\frac{11}{3},\,1;\,\frac{17}{6},\,\frac{17}{6};\,Z\,\frac{1-\delta^2}{\delta}\right)\Big\}.\end{aligned}\tag{27}$$

As $\rho = 0$, Eq. (27) becomes

$$b_{\chi_1\chi_2}(0)=\frac{B_{\chi_1\chi_2}(0)}{\sqrt{B_{\chi_1}(0)B_{\chi_2}(0)}}=\frac{1-\delta^{5/6}}{(1-\delta^2)^{5/12}}.\tag{28}$$



Acording to [8, 9], this is the normalized frequency-correlation coefficient for the amplitude fluctuations at a given point for the case of plane waves. This is a particularly interesting result, since it implies that we can use (28) for a beam. The accompanying figure shows a plot of the function $b_{\chi_1\chi_2}(\rho)$, $(\Delta\lambda=\lambda_1-\lambda_2)$.

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REFERENCES

1. M. F. Bakhareva, Radiotekhnika i elektronika, No. 1, 1959; No. 10, 1961; No. 3, 1968; No. 6, 1968.
2. S. Ya. Braude and E. A. Kaner, Izv. vuzov, radiofizika, No. 2, 1962.
3. L. M. Erukhimov, Geomagnetizm i aeronomiya, No. 1, 1964; No. 2, 1966.
4. V. A. Alimov and L. M. Erukhimov, Izv. vuzov, radiofizika [Radiophysics and Quantum Electronics], No. 5, 1967; No. 2, 1968.
5. Ya. I. Al'ber, L. M. Erukhimov, V. A. Ryzhov, and V. P. Uryadov, Izv. vuzov, radiofizika [Radiophysics and Quantum Electronics], No. 9, 1968.
6. L. M. Erukhimov and V. P. Uryadov, Izv. vuzov, radiofizika [Radiophysics and Quantum Electronics], No. 12, 1968.
7. V. I. Tatarskii and L. N. Zhukova, DAN SSSR, 124, No. 3, 1959.
8. V. A. Eliseevnin, Vestn. Mosk. un-ta, fiz., astron. [Moscow University Physics Bulletin], No. 4, 1969.
9. V. I. Tatarskii, Wave Propagation in the Turbulent Atmosphere [in Russian], Nauka, Moscow, 1967.
10. I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Sums, Series, and Products [in Russian], Fizmatgiz, Moscow, 1962.
11. D. L. Fried, JOSA, 57, No. 2.

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