# ROOT-TRAJECTORY STUDY OF TWO-DIMENSIONAL SYSTEMS HAVING ANTISYMMETRIC CROSS COUPLING

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The basic properties of the root trajectories, which can be used for qualitative and quantitative of the dynamics of identical two-dimensional systems, are studied for characteristic equations of the form $N(p)+jaM(p)=0$ with complex coefficients. Geometric and analytic methods for constructing these root trajectories are described. The degree of the trajectory equation in $\omega^2$ is much lower than that of the characteristic equation, so it becomes easier to analyze two-channel systems described by high-order differential equations. Illustrative constructions of root trajectories are described.

In this paper we will study the properties of two-dimensional systems having identical channels and antisymmetric cross (forward*) coupling (Fig. 1) by the root-trajectory method [1, 2]. The free parameter (the trajectory parameter), which varies over a wide range, is chosen to be the gain $a$ of the coupling units. The characteristic equations for antisymmetric-coupling systems can be written in the form of factors having complex coefficients, so a study of the properties of two-dimensional systems reduces to a study of the general properties of the trajectories of the roots of the characteristic equations having complex coefficients. Using the basic properties of the root trajectories, the equations of these trajectories, and the equations of trajectory parameter, we can qualitatively and quantitatively study high-order two-dimensional systems.

The characteristic equations for two-dimensional systems with identical channels and antisymmetric cross coupling are [2, 3]

$$[1+W(p)]^2+[a(p)]^2=[1+W(p)+ja(p)][1+W(p)-ja(p)]=0, \tag{1}$$

$$W(p)=\frac{k\Psi_m(p)}{\Phi_n(p)}, \qquad (n\geqslant m) \tag{2}$$

where the transfer function of the primary channel,

$$a(p)=\frac{aA_l(p)}{B_s(p)}, \tag{3}$$

is the transfer function for the cross-coupling unit. In the transfer function, p is the Laplace variable, and $\Phi_n(p)$, $\Psi_m(p)$, $B_s(p)$ and $A_l(p)$ are polynomials in p, of positive integral degree n, m, s and $\ell$, respectively with real coefficients.

Using Eqs. (2) and (3), we write characteristic equation (1) as

$$[B_s(\Phi_n+k\Psi_m)+jaA_l\Phi_n][B_s(\Phi_n+k\Psi_m)-jaA_l\Phi_n]=0. \tag{4}$$

In the characteristic equation, $p=\delta+j\omega$ is the complex frequency. We will adopt as a parameter of the root trajectories the gain $a$ of the class-coupling unit; this parameter varies over a range $0<a<\infty$.

It is sufficient to study and construct the root trajectories for one of the factors in Eq. (4), e.g., the first,

$$[B_s(\Phi_n+k\Psi_m)+jaA_l\Phi_n]=0, \tag{5}$$

since the roots of the other factor are simply the complex conjugates. In other words, we will apply plus and minus signs to the free parameter $a$ in Eq. (4) and treat it as the root-trajectory parameter, which varies over the range $-\infty<a<\infty$. The two factors in Eq. (4) correspond to positive $(a\geqslant 0)$ and negative $(a\leqslant 0)$ root hodographs or, correspondingly, to odd and even trajectories for the roots of Eq. (5), which constitute the complete root hodograph of the system. The initial and terminal points are double points.

We turn to the geometric method for plotting the root trajectories, which brings out several features of the root trajectories for equations of the form (4) having complex coefficients; in addition, this method can be used in a qualitative analysis in the properties of systems having antisymmetric cross coupling.

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*Systems with reverse cross coupling can be converted into systems for forward cross coupling.

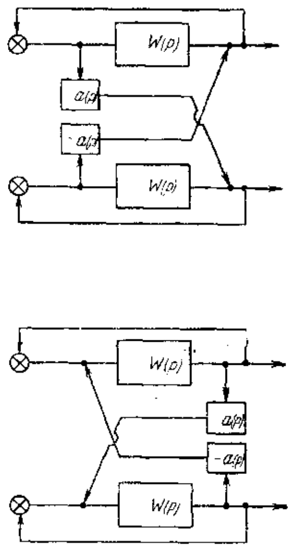


Fig. 1

## GEOMETRIC METHOD FOR CONSTRUCTING ROOT TRAJECTORIES

In Eq. (5) we assume the coefficients of the polynomials and the gain k of the primary channel to be specified. The degree (n + s) of the first term in Eq. (5) is governed by the number of poles in the primary-channel and cross-coupling units; the degree of the second term (n + ℓ) is governed by the number of poles of the transfer function of the primary channel and by the number of zeros of the transfer function of the cross-coupling channel. The number n + s also corresponds to the number of initial trajectory points ($a$ = 0),

$$B_s(\Phi_n + k\Psi_m) = 0 \tag{6}$$

(of which some can be determined from k); and n + ℓ corresponds to the number of terminal points ($a = \pm\infty$)

$$A_l \Phi_n = 0. \tag{7}$$

The number of primary(initial and terminal) trajectory points determines the class of the system. In this case, Eq. (5) is the equation of class $[n+s;\ n+l]$. We note that the complete characteristic equation, (1), is an equation of class $[2(n+s);\ 2(n+l)]$.

As $a$ varies continuously from 0 to $\pm\infty$, $2(n+s)$ root trajectories leave the initial $p_\nu$ trajectories [ $\nu = 1, 2, \ldots (n+s)$ ]; of them, 2(n + $l$) end at terminal points $z_\mu$ [ $\mu = 1, 2, \ldots (n+l)$ ], while the other 2(s − $l$) leave at infinity (for $s > l$). $2(s-l)$, these trajectories form the complete root hodograph of Eq. (4). In addition, the positive ($a \geqslant 0$) and negative root hodographs of Eq. (5) form the complete root hodograph of system (4). Knowing the initial points $p_\nu$ and the terminal points $z_\mu$, we can rewrite Eq. (5) as

$$c_0 \prod_1^{n+s} (p - p_\nu) + jad_0 \prod_1^{n+l} (p - z_\mu) = 0, \quad n > m. \tag{8}$$

For points on trajectories p = p* Eq. (8) becomes an identity.

In the p plane, we write the vectors $(p^* - p_\nu)$ and $(p^* - z_\mu)$ as

$$(p^* - p_\nu) = |p^* - p_\nu| e^{j\varphi}, \ (p^* - z_\mu) = |p^* - z_\mu| e^{j\psi}, \tag{9}$$

where $|p^* - p_\nu|$, $|p^* - z_\mu|$ are the moduli and $\varphi$ and $\psi$ are the phases of these vectors (in the complex plane, these are the angles made by the vectors and the positive real axis). Using Eq. (9), we find from Eq. (8) that

$$\bar{a} = -\frac{\prod_1^{n+s} |p^* - p_\nu|}{\prod_1^{n+l} |p^* - z_\mu|} e^{j\left(\sum_1^{n+s} \varphi_\nu - \sum_1^{n+l} \psi_\mu - \frac{\pi}{2}\right)},$$

where $\bar{a} = \frac{ad_0}{c_0}$ is the reduced trajectory parameter (with n = m, we have $\bar{a} = \frac{aa_0 d_0}{c_0(a_0 + kb_0)}$ ). From the condition that the reduced trajectory parameter be real, we find the basic phase equation:

$$\sum_1^{n+s} \varphi_\nu - \sum_1^{n+l} \psi_\mu - \frac{\pi}{2} = N\pi, \qquad N = 0, \pm 1, \ldots . \tag{10}$$

Using Eq. (10), we can rewrite the equation for the root-trajectory as

$$\bar{a} = (-1)^{N+1} \frac{\prod_1^{n+s} |p^* - p_\nu|}{\prod_1^{n+l} |p^* - z_\mu|}, \qquad N = 0, \pm 1, \ldots . \tag{11}$$

The odd values of N ($\bar{a} \geqslant 0$) correspond to odd trajectories, and the even N ($\bar{a} \leqslant 0$) correspond to even trajectories.

The asymptotic properties of the root hodographs for equations having complex coefficients are found by analogy with the case of equations having real coefficients [1]. The center of the asymptote star is found from

$$a^* = \frac{\sum_1^{n+s} p_\nu - \sum_1^{n+l} z_\mu}{(n+s) - (n+l)} = \frac{\sum_1^{n+s} p_\nu - \sum_1^{n+l} z_\mu}{(s-l)}. \tag{12}$$

The angle between the asymptote and the real axis are found from the basic phase equation. For infinitely remote points from the trajectories, $\varphi_\nu$ and $\Psi_\mu$ in Eq. (10) are

$$\varphi_N = \frac{\left(N+\frac{1}{2}\right)\pi}{s-l}, \qquad N=0,\ \pm 1,\dots . \tag{13}$$

The direction of asymptote (13) is thus governed only by the number of poles s and by the number of zeros ℓ of the transfer function for the coupling unit. With s = $l$, the roots from 2(n + s) initial points follow 2(n + s) trajectories to 2(n + s) terminal (end) points as $a \to \pm\infty$. There are no asymptotes; the center of the asymptotes is at infinity. If $s=l=0$, i.e., if the cross coupling is rigid, $a$(p) = $a$, there are again no asymptotes, and the asymptote center is at infinity. If $s<l$, we adopt ρ = 1/$a$ as the trajectory parameter in Eq. (5), so that initial and terminal points are interchanged, and the asymptote star is rotated in the direction opposite to that of (13).

The fundamental difference between these root hodographs and those corresponding to characteristic equations having real coefficients is that the asymptote star is rotated through an angle $\varphi_0=\pi/2(s-l)$ about the positive real axis. It follows that the real axis never coincides with the asymptotes of the root hodographs for equations of type (4).

Analysis of basic phase equations (10) shows that it does not hold for points on the real axis other than the real basic points, since for these points N cannot be equal to an integer. Accordingly, roots cannot move along the real axis from initial points on the axis.

We will show below that the root trajectory can intersect the real axis only at the basic points on this axis. Therefore, a characteristic equation of the form (5) with complex coefficients and thus Eq. (1) for any $a$ have only oscillatory solutions; they do not have solutions of the form $e^{-\delta t}$.

We can give a qualitative idea of the root hodographs possible for system (1) if we know the positions of the basic points on the p plane, by exploiting the asymptotic properties of the root trajectories (the position of the center of the asymptote star and the direction of the asymptotes) and by using the basic basic phase equation to determine the angles at which the trajectories leave the initial points and at which they arrive at the terminal point [1].

We can construct root hodographs for equations of type (4) of any order and having complex coefficients, and we can "calibrate" these hodographs in terms of the $a$ values, by measuring the arguments in (10) and the moduli in (11) of vectors drawn from the basic points to given points on the complex-frequency plane.

## ANALYTIC METHOD FOR CONSTRUCTING THE ROOT TRAJECTORIES

We turn now to an analytic derivation of the equations of the root trajectories and of the equations for the trajectory parameter. Using

$$N(p) = B_s(\Phi_n + k\Psi_m), \ M(p) = A_l\Phi_n, \tag{14}$$

we can convert (5) to

$$N(p) + jaM(p) = 0. \tag{15}$$

Here N(p) and M(p) are polynomials of degree $(n+s)$ and $(n+l)$, respectively. Substituting $p=\delta+j\omega$ into (14) and separating and real and imaginary parts, we find

$$N = N_r + j\omega N_j, \quad M = M_r + j\omega M_j, \tag{16}$$

where

$$N_r = N(\delta) - \frac{\omega^2}{2!}N''(\delta) + \frac{\omega^4}{4!}N^{IV}(\delta) - \dots ,$$

$$N_j = N'(\delta) - \frac{\omega^2}{3!}N'''(\delta) + \frac{\omega^4}{5!}N^{V}(\delta) - \dots ,$$

$$M_r = M(\delta) - \frac{\omega^2}{2!}M''(\delta) + \frac{\omega^4}{4!}M^{IV}(\delta) - \dots ,$$

$$M_j = M'(\delta) - \frac{\omega^2}{3!}M'''(\delta) + \frac{\omega^4}{5!}M^{V}(\delta) - \dots . \tag{17}$$

Using (16) into (15), and equating the real and imaginary parts to zero, we find two equations for the three quantities δ, ω, and $a$(all other quantities are specified):

$$N_r - a\omega M_j = 0, \quad \omega N_j + aM_r = 0. \tag{18}$$

We then find the root-trajectory equation, which relates to δ and ω,

$$\begin{vmatrix} N_r & -\omega M_j \\ \omega N_j & M_r \end{vmatrix} = N_rM_r + \omega^2 N_jM_j = 0, \tag{19}$$

and we find equations for the trajectory parameter

$$a = \frac{N_r}{\omega M_j} = -\frac{\omega N_j}{M_r}. \tag{20}$$

It follows from (18) that the real axis is generally not a root trajectory, as in the case of characteristic equations having real coefficients [1]. Trajectory equation (19) is a complete root portrait of system (4), with the odd trajectories corresponding to the first factor and the even trajectories to the second.

Using (17), and collecting terms of identical degree in $\omega$, we convert from (19) to

$$NM - \omega^2\left(\frac{NM''}{0!2!} - \frac{N'M'}{1!1!} + \frac{N''M}{2!0!}\right) + \omega^4\left(\frac{NM^{IV}}{0!4!} - \frac{N'M'''}{1!3!} + \frac{N''M''}{2!2!} - \frac{N'''M'}{3!1!} + \frac{N^{IV}M}{4!0!}\right) - \omega^6\left(\frac{NM^{VI}}{0!6!} - \ldots\right) + \ldots = 0. \tag{21}$$

The differentiation of the functions $N(\delta)$ and $M(\delta)$ indicated with respect to $\delta$. Setting $\omega = 0$ into Eq. (21), and using (14), we find an equation for the intersection of the root trajectories and the real axis:

$$N(\delta)M(\delta) = B_s(\delta)[\Phi_n(\delta) + k\Psi_m(\delta)]A_l(\delta)\Phi_n(\delta) = 0.$$

These points may be either initial or terminal points on the real axis; there are no longer other intersections of the root trajectories in the real axis or the $a$ range (from 0 to $\pm\infty$).

We find an equation for multiple points by using Eq. (15) and the derivative of this equation with respect to $p$:

$$\begin{vmatrix} N(p) & M(p) \\ N'(p) & M'(p) \end{vmatrix} = NM' - N'M = 0. \tag{22}$$

It follows that the positive and real hodographs can have $(2n+s+l-1)$ 1-fold points. They can fall on the real axis, however, only if they coincide with real basic points. Root-trajectory equation (21) does not coincide with multiple-point equation (22) for $\omega = 0$, as in the case of equations of the form $\Phi_n + k\Psi_m = 0$ having real coefficients [1].

The critical frequencies $\omega_k$ and the critical $a_k$ are found from root-trajectory equation (19) or (21) and from (20) for the case $\delta = 0$.

Only even powers of $\omega$ appear in (21), so this equation is of a much lower degree in $\omega^2$ than the characteristic equation. For example, for equations of classes [1; 1], [2; 0], [2; 1] and [3; 0], the root-trajectory equations are linear in $\omega^2$; for equations of classes [2; 2], [3; 1], [3; 2], [4; 0], and [5; 0], these equations are of second degree in $\omega^2$; and for equations of classes [3; 3], [4; 2], [4; 3], [5; 1], [5; 2], [6; 0], [6; 1], and [7; 0], these equations are of third degree in $\omega^2$.

In using this method to study characteristic equations of relatively high degree, we can therefore study root-trajectory equations of significantly lower degree, which, because the initial equation, (1), is written as a product of two factors, and second, because the root-trajectory equation contains only even powers of $\omega$. The root trajectories for characteristic equations of the form (4) of high degree can be constructed in a simple manner because the coupling-unit gain $a$ is used as a trajectory parameter. When gain k of the basic unit is used as a parameter in (4), the trajectory equation will contain both even and hard powers of $\omega$ [3, 4].

Other properties of the root trajectories will be similar to those corresponding to characteristic equations having real coefficients [1]. By qualitatively analyzing the root hodographs, we can determine the regions in the p plane where the trajectories must be plotted exactly on the basis of Eq. (21).

We will now construct some root hodographs for systems of various classes to illustrate the basic properties of the trajectories for characteristic equations with complex coefficients when $a$ is used as a trajectory parameter in (4). The initial and terminal trajectory points are shown by crosses and circles. Double initial and terminal points correspond to the complete root hodograph. Odd trajectories are shown by a single error, and even trajectories are shown by a double error.

Let us assume we have a characteristic equation of class [1; 1],

$$(p+1) + jap = 0,$$

where $0 \leqslant a < \infty$. Odd trajectories (Fig. 2) leave initial point $p_1 = -1$ and at terminal point $z_1 = 0$. Even trajectories correspond to the conjugate equation $(p+1) - jap = 0$, where $-\infty < a \leqslant 0$. Equation (21) becomes

$$\delta(\delta+1) + \omega^2 = 0.$$

We use this equation to construct the odd and even trajectories, i.e., the complete root hodograph of the system. From (20), we find equations for $a$ for the odd trajectories,

$$a = \frac{\delta+1}{\omega} = -\frac{\omega}{\delta},$$

and for the even trajectories

$$-a=\frac{\delta+1}{\omega}=-\frac{\omega}{\delta}.$$

There are no asymptotes. The center of the asymptotes is at infinity.

The complete root portrait of the system having a characteristic equation $(p+1)^2+a^2p^2=0$ (Fig. 2) thus shows that the roots move along a circle of radius 0.5 centered on the point (0, −0.5) and that the roots do not enter the right half-plane for any $a$. The system is stable for all values of all coupling-unit gain.

Let us consider a characteristic equation of class [2; 1] (the system is described by a first-order differential equation):

$$(p+1)(p+2)+ja(p+3)=0 \quad \text{(see Fig. 3)}.$$

Since we are interested in the complete root hodograph, we assume $-\infty<a<+\infty$. The initial trajectory points, $p_1=-1$ and $p_2=-2$, and the terminal point, $z_1=-3$, are double. The root-trajectory equation is found from Eq. (21): $\omega^2+(\delta+1)(\delta+2)(\delta+3)=0$. Equation (20) becomes

$$a=\frac{\delta^2+3\delta}{\omega}+2-\omega^2=-\frac{\omega(2\delta+3)}{\delta+3}.$$

Figure 3 shows the root trajectories. The center of the asymptote star is at point $a^*=0$. The asymptotes make angles of $\varphi_0=\pi/2$ and $\varphi_1=3\pi/2$ with the real axis. Roots from the double initial point $p_1=-1$ approach the asymptote as they go to infinity. Roots from the initial point $p_2=-2$ follow even and off trajectories to the double terminal point $z_1=-3$. The root portrait is completely in the left-half plane. This system is stable for any $a$.

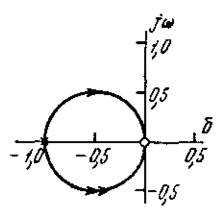


Fig. 2

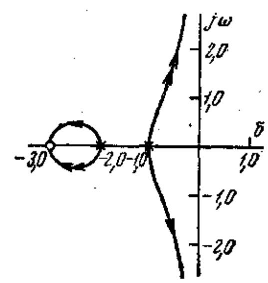


Fig. 3

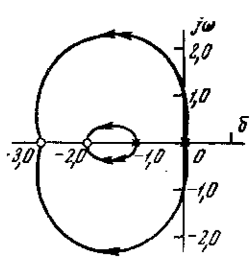


Fig. 4

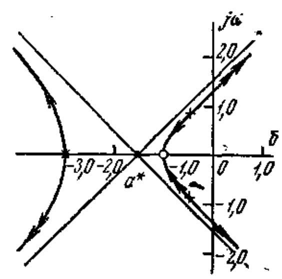


Fig. 5

For a characteristic equation of class [2; 2],

$$p(p+1)+ja(p+2)(p+3)=0 \quad \text{(see Fig. 4)},$$

where $-\infty<a<\infty$, the initial points $p_1=0$ $p_2=-1$, and the terminal points $z_1=-2$ and $z_2=-3$, are double points. From (21) we find the root-trajectory equation to be

$$\omega^4+(2\delta^2+6\delta+1)\omega^2+\delta(\delta+1)(\delta+2)(\delta+3)=0, \tag{23}$$

and from (20) we find the equation for $a$:

$$a=\frac{\delta^2+\delta-\omega^2}{\omega(2\delta+5)}=-\frac{\omega(2\delta+1)}{\delta^2+5\delta+6-\omega^2}. \tag{24}$$

Figure 4 shows the root trajectories. The center of the asymptote star is at infinity. With $\delta=0$, we find from Eqs. (23) and (24) the critical frequencies and trajectory parameters, respectively: $\omega_k=\pm1$, $a_k=\mp0{,}2$. The system is stable in the region $0{,}2<a<\infty$. The system is unstable for any $a$ values in the range $0<a<0{,}2$.

For an equation of class [3: 1] (Fig. 5), in which case the system is described by a sixth-order differential equation, we have

$$(p^3+4p^2+4p+3)+ja(p+1)=0 \quad \text{(see Fig. 5)}.$$

The $a$ range is $-\infty<a<\infty$. The initial point, $p_{1,2}=-0{,}5\pm0{,}866j$, $p_3=-3$, and the terminal points $z_1=-1$, are double points. The center of the asymptote star is at $a^*=-1{,}5$ (Fig. 5). The asymptotes makes angles of $\varphi_0=45°$, $\varphi_1=135°$, $\varphi_2=225°$, and $\varphi_3=315°$. with respect to the real axis. From Eq. (21) and (20) we find the root-trajectory equation and

$$\omega^4-\delta\omega^2-(\delta+1)(\delta+3)(\delta^2+\delta+1)=0, \quad (25)$$

$$a=\frac{\delta^3+4\delta^2+4\delta+3-(3\delta+4)\omega^2}{\omega}=-\frac{(3\delta^2+8\delta+4)\omega-\omega^3}{\delta+1}. \quad (26)$$

Setting $\delta$ = 0 in Eqs. (25) and (26), we find the critical frequencies and the corresponding critical $a$ values: $\omega_k=\pm1{,}315$ and $a_k=2{,}98$. The system is stable for $0<a<2{,}98$; with $a>2{,}98$, the roots enter the right half-plane, and this system becomes unstable.

We see from these examples that by studying the root trajectories we can follow the changes in the dynamic properties of two-channel systems which antisymmetric cross coupling which is described by differential equations of quite high order: this study can be carried out in the stability and at its boundaries.

Two-dimensional systems with antisymmetric reverse cross coupling could be studied in a similar manner.

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