SECOND-ORDER SECULAR PERTURBATIONS IN THE MOTION OF A REMOTE SATELLITE

V. P. Dolgachev

Vestnik Moskovskogo Universiteta. Fizika, Vol. 25, No. 3, pp. 320-28, 1970

UDC 521.13

Equations are derived for the second-order secular perturbations in the orbital elements for remote earth satellites under the condition that the perturbing function contains terms of third degree in the Lagrange variables.

In this paper we study orbital evolution of an earth satellite with an account of the flattening of the earth and of the gravitational attraction of the moon, which is assumed to be a mass point. We previous derived [1] analytic expressions for the second-order secular perturbations in the orbital elements of remote satellites: the perturbing function in [1] contained terms of up to second degree in the Lagrange variables. Below, we use the Lyapunov method to find a general solution of the nonlinear system of differential equations for the secular perturbations, and we briefly analyze the results.

§1. FORMULATION OF THE PROBLEM

We seek the secular perturbations in the orbital elements of remote earth satellites under the condition that the longitude Ω_L of the ascending node and the longitude ω_L of the perihelion of the lunar orbit vary secularly in accordance with the Hill-Brown theory.

We showed in [1] that if we retain in the doubly-averaged function R resulting from the lunar attraction and the flattening of the earth, terms of up to second degree in the Lagrange variables, we find

$$R_2 = a_1 \sin \Omega_L p + q(a_1 \cos \Omega_L - a_2) - a_3 p^2 - a_4 p^2 + a_5 (h^2 + k^2),$$

where

$$a_1 = \gamma \sin i_L,\ a_2 = \beta \sin 2\varepsilon,\ \gamma = \frac{3}{2} n_1^2 F_1 a^2,$$

$$a_3 = \gamma \cos^2 i_L + 2\beta \cos^2 \varepsilon,\ a_4 = \gamma \cos^2 i_L + 2\beta \cos 2\varepsilon,$$

$$a_5 = \frac{1}{4}\gamma + \frac{1}{2}\beta\left(1 - \frac{3}{2}\sin^2 \varepsilon\right),\quad \beta = Jn^2 a_0^2,$$

$$n_1^2 = \frac{\mu_L}{a_L^3},\quad F_1 = 1 + \frac{3}{2} e_L^2 + \frac{15}{8} e_L^4 + \frac{35}{16} e_L^6, \tag{1}$$

a_L, e_L, i_L, and Ω_L are the Kepler orbital elements of the moon; J is the flattening parameter; a_0 is the equatorial radius of the earth; $\mu_L = fm_L$, ; f is the gravitational constant; m_L is the lunar mass; and ε is the inclination of the equatorial plane with respect to the ecliptic plane.

The Lagrange variables are defined by

$$h = e \sin \pi,\quad p = \sin \frac{i}{2} \sin \Omega,$$

$$k = e \cos \pi,\quad q = \sin \frac{i}{2} \cos \Omega, \tag{2}$$

where a, e, i, Ω, and ω the Kepler ecliptic elements of the satellite's orbit; $\pi = \omega + \Omega$; and n represents the average satellite motion. The differential equations of motion of the satellite in terms of elements (2) are

$$\frac{dh}{dt} = \frac{\sqrt{1-e^2}}{na^2} \frac{\partial R}{\partial k} + \frac{1}{2} \frac{k}{na^2 \sqrt{1-e^2}} \left(p \frac{\partial R}{\partial p} + q \frac{\partial R}{\partial q}\right),$$

$$\frac{dk}{dt} = -\frac{\sqrt{1-e^2}}{na^2} \frac{\partial R}{\partial h} - \frac{1}{2} \frac{h}{na^2 \sqrt{1-e^2}} \left(p \frac{\partial R}{\partial p} + q \frac{\partial R}{\partial q}\right),$$

$$\frac{dp}{dt} = \frac{1}{4na^2 \sqrt{1-e^2}} \frac{\partial R}{\partial q} - \frac{1}{2} \frac{p}{na^2 \sqrt{1-e^2}} \left(k \frac{\partial R}{\partial h} - h \frac{\partial R}{\partial k}\right),$$

$$\frac{dq}{dt} = -\frac{1}{4na^2 \sqrt{1-e^2}} \frac{\partial R}{\partial p} - \frac{1}{2} \frac{q}{na^2 \sqrt{1-e^2}} \left(k \frac{\partial R}{\partial h} - h \frac{\partial R}{\partial k}\right). \tag{3}$$

Using R_2 as R, and substituting R_2 into (3), we find a linear inhomogeneous system of differential equations. Its solution is [1]

$$
\begin{gathered}
h = C_1 \cos bt + C_2 \sin bt, \quad k = C_2 \cos bt - C_1 \sin bt, \\
p = A_1 \sin \Omega_L - b_2 (C_3 \cos st + C_4 \sin st), \\
q = B_0 + B_1 \cos \Omega_L + s (C_4 \cos st - C_3 \sin st),
\end{gathered} \tag{4}
$$

where

$$
\begin{gathered}
b = \frac{2a_5}{na^2}, \qquad A_1 = \frac{b_1 (\alpha - b_2)}{\alpha^2 - s^2}, \\
\Omega_L = \alpha t + \Omega_{0L}, \qquad b_2 = \frac{a_4}{2na^2}, \qquad s^2 = b_2 b_3, \\
b_3 = \frac{a_3}{2na^2}, \quad B_1 = \frac{b_1 (\alpha - b_3)}{\alpha^2 - s^2}, \quad B_0 = -\frac{b_0}{b_2}, \\
b_0 = \frac{a_2}{4na^2}, \qquad b_1 = \frac{a_1}{4na^2}, \qquad \alpha = -19{,}3411 \text{ deg/yr}
\end{gathered}
$$

Ω_{0L} is the initial longitude of the ascending node of the lunar orbit, and C_i $(i=1, 2, 3, 4)$ are arbitrary integration constants. Equations (4) shows that the variables h and k, proportional to the eccentricity of the satellite orbit, do not depend on Ω_L under these conditions. From knowledge of the changes in e over long time intervals we can draw definite conclusions about the time the satellite will spend in orbit. For this reason, equations are obtained for the behavior of the orbital elements under the condition that terms of third degree in h, k, p and q are retained in the expression for the perturbing function.

§2. DIFFERENTIAL EQUATIONS OF MOTION AND METHOD FOR SOLVING THEM

Singling out the terms of third degree in variables (2) in the expression for the perturbing function for this problem, we find

$$
\begin{gathered}
R_3 = \frac{1}{2} \beta \sin 2\varepsilon q \left[5 (p^2 + q^2) - 3 (h^2 + k^2) \right] + \\
+ \gamma \sin i_L \left[\frac{1}{2} (p \sin \Omega_L + q \cos \Omega_L)(13h^2 - 7k^2 - 5p^2 - 5q^2) + \right. \\
\left. + 10hk (p \cos \Omega_L - q \sin \Omega_L) \right], \quad R = R_2 + R_3.
\end{gathered}
$$

The differential equations of motion in form (3) are

$$
\begin{gathered}
\frac{dh}{dt} = bk - 14b_0 kq - 26b_1 k (p \sin \Omega_L + q \cos \Omega_L) + \\
+ 40b_1 h (p \cos \Omega_L - q \sin \Omega_L), \\
\frac{dk}{dt} = -bh + 10b_0 hq + 50b_1 h (p \sin \Omega_L + q \cos \Omega_L) + \\
+ 40b_1 k (q \sin \Omega_L - p \cos \Omega_L), \\
\frac{dp}{dt} = -b_0 + b_1 \cos \Omega_L - b_2 q + \frac{1}{2} b_0 (5p^2 + 10q^2 - 3h^2 - 3k^2) + \\
+ \frac{1}{2} b_1 \cos \Omega_L (13h^2 - 7k^2 - 5p^2 - 15q^2) - 5b_1 \sin \Omega_L (2hk + pq), \\
\frac{dq}{dt} = -b_1 \sin \Omega_L + b_3 p - 5b_0 pq + 5b_1 \cos \Omega_L (pq - 2hk) + \\
+ \frac{1}{2} b_1 \sin \Omega_L (5q^2 + 15p^2 - 13h^2 + 7k^2).
\end{gathered} \tag{5}
$$

It has been shown [2] that the nonlinear system of equations for the first-order secular perturbations in the lunar-orbit coordinate systems has the particular solution $q^* = \text{const}$, $h = k = p = 0$. Systems (5) also has a particular solution of this type provided that $\Omega_L = \text{const}$ and that

$$
|\Omega_L| \leqslant 126°{,}836 \ (q^* > 0) \text{ and } |\Omega_L| \geqslant 51°{,}686, \ (q^* < 0).
$$

In order to integrate system (5) by the Lyapunov method, we must replace variables according to

$$
h = h_1 + h_2, \quad k = k_1 + k_2, \quad p = p_1 + p_2, \quad q = q_1 + q_2, \tag{6}
$$

where h_1, k_1, p_1, and q_1 are given by Eqs. (4) and h_2, k_2, p_2, and q_2 are some functions of the time, which are to be determined. After transformation (6), the right sides of system (5) become a set of linear and quadratic terms in h_i, k_i, p_i, and q_i $(i=1, 2)$, which vanish when $h_i = k_i = p_i = q_i = 0$. According to the Lyapunov theorem [3], the solution of system (5) can be written as absolutely converging series expansions in powers of C_j $(j=1, 2, 3, 4)$. Below we will obtain only those terms of these series which are of up to second degree in the C_j.

§3. EQUATIONS FOR THE PERTURBATIONS

Carrying out transformation (6), and replacing h_1, k_1, p_1 and q_1 by their expressions in (4), we find a linear inhomogeneous system of differential equations for h_2, k_2, p_2, and q_2.

The right sides of this system are quite lengthly expressions, that we omit for brevity. We merely state that the inhomogeneous part of the variables h_2 and k_2 contain sine and cosine functions with frequencies b, $b\pm\alpha$, $b\pm s$, $2\alpha\pm b$, and $b\pm s\pm\alpha$, for the inhomogeneous part of the equations for p_2 and q_2 consists of similar functions with frequencies α, 2α, 3α, s, $2s$, $\alpha\pm s$, $\alpha\pm 2s$, $2\alpha\pm s$, and $\alpha-2b$.

It should be noted that the frequencies b and s are resonant frequencies. The solution for the variables h_2 and k_2 can be written as

$$h_2 = \sum_{i=-1}^{1} \sum_{j=-2}^{2} M_{ij} \sin [(b+is)t + j\Omega_L] + N_{ij} \cos [(b+is)t + j\Omega_L] + $$
$$+ t(C_1\varkappa \sin bt - C_2\varkappa \cos bt),$$
$$k_2 = \sum_{i=-1}^{1} \sum_{j=-2}^{2} \widetilde{M}_{ij} \sin [(b+is)t + j\Omega_L] + \widetilde{N}_{ij} \cos [(b+is)t + j\Omega_L] + $$
$$+ t(-C_1\varkappa \sin bt + C_2\varkappa \cos bt),$$

where the indices ± 1, ± 2 have been omitted, the coefficients are given in Tables 1 and 2, and

$$C_{14} = C_1C_4 + C_2C_3, \quad \overline{C}_{14} = C_1C_4 - C_2C_3, \quad C_{24} = C_2C_4 + C_1C_3,$$
$$\overline{C}_{24} = C_2C_4 - C_1C_3, \quad \varkappa = 12b_0B_0 + 19b_1(A_1 + B_1).$$

The solution for the variables p_2 and q_2 can be written

$$p_2 = \sum_{i=-1}^{2} \sum_{j=0}^{3} G_{ij} \sin (ist + j\Omega_L) + H_{ij} \cos (ist + j\Omega_L) + $$
$$+ f_1 [(C_2^2 - C_1^2) \sin (2bt - \Omega_L) + 2C_1C_2 \cos (2bt - \Omega_L)] + $$
$$+ \frac{b_2}{s} t(-B_3 \sin st + B_2 \cos st),$$
$$q_2 = \sum_{i=-1}^{2} \sum_{j=0}^{3} \widetilde{G}_{ij} \sin (ist + j\Omega_L) + \widetilde{H}_{ij} \cos (ist + j\Omega_L) + $$
$$+ f_2 [2C_1C_2 \sin (2bt - \Omega_L) + (C_1^2 - C_2^2) \cos (2bt - \Omega_L)] + A_{00} + $$
$$+ t(B_2 \sin st + B_3 \cos st).$$

In the double sum for p_2 and q_2 the indices -10, ± 13, 22, and 23 have been omitted. The exclusive expressions for the coefficients in the equations for p_2 and q_2 are

$$G_{01} = \frac{1}{s^2 - \alpha^2} \left\{ \lambda_{01}\left(\alpha - \frac{1}{2} b_2\right) + 5b_0B_0 (A_1b_2 - 2\alpha B_1) - \mu_{01} (b_2 - \alpha) - b_2\nu_{01} \right\},$$
$$\widetilde{H}_{01} = \frac{1}{s^2 - \alpha^2} \left\{ \lambda_{01}\left(\frac{1}{2}\alpha - b_3\right) + 5b_0B_0 (2b_3B_1 - \alpha A_1) + \mu_{01} (b_3 - \alpha) + \alpha\nu_{01} \right\},$$
$$\lambda_{01} = \frac{5}{8} b_1 [2(C_3^2 + C_4^2)(b_2^2 + 3s^2) + (A_1 + B_1)^2 + 12B_0^2 + 8B_1^2],$$
$$H_{01} = \widetilde{G}_{01} = 0, \quad \mu_{01} = \frac{3}{2} b_1 (C_1^2 + C_2^2), \quad \nu_{01} = \frac{5}{8} b_1 (A_1^2 + 2A_1B_1 - 3B_1^2),$$
$$G_{02} = \frac{5}{s^2 - 4\alpha^2} \left[\lambda_{02}\left(\alpha - \frac{1}{2} b_2\right) - \mu_{02} + \alpha\nu_{02} + \frac{1}{2} b_0b_2A_1B_1 \right],$$
$$\widetilde{H}_{02} = \frac{5}{s^2 - 4\alpha^2} \left[\lambda_{02}\left(\alpha - \frac{1}{2} b_3\right) + \mu_{02} - \frac{1}{2} b_3\nu_{02} - \alpha b_0A_1B_1 \right],$$
$$H_{02} = \widetilde{G}_{02} = 0, \quad \lambda_{02} = 3b_1B_0B_1, \quad \mu_{02} = b_1A_1B_0 \left(\alpha + \frac{1}{2} b_2\right),$$
$$\nu_{02} = b_0 \left(\frac{1}{2} A_1^2 - B_1^2\right), \quad H_{03} = \widetilde{G}_{03} = 0,$$
$$G_{03} = \lambda_{03} [\mu_{03} (3\alpha - b_2) - \nu_{03} (3\alpha + b_2)],$$
$$\widetilde{H}_{03} = \lambda_{03} [\mu_{03} (3\alpha - b_3) + \nu_{03} (3\alpha + b_3)],$$
$$\mu_{03} = 3B_1^2 - A_1^2, \quad \lambda_{03} = \frac{5b_1}{8(s^2 - 9\alpha^2)}, \quad \nu_{03} = 2b_1A_1B_1,$$

$$G_{20} = -\lambda_{20}(C_3^2 - C_4^2),\ H_{20} = 2\lambda_{20}C_1C_2,\ \lambda_{20} = \frac{5}{3}b_0 s,$$

$$\tilde{G}_{20} = \frac{\lambda_{20}}{s}\mu_{20}C_3C_4,\ \tilde{H}_{20} = \frac{\lambda_{20}}{2s}\mu_{20}(C_3^2 - C_4^2),\ \mu_{20} = b_3 + \frac{3}{2}b_2,$$

$$G_{k1} = -C_4\lambda_{k1},\ H_{k1} = -kC_3\lambda_{k1},\ \tilde{G}_{k1} = -C_3\mu_{k1},\ \tilde{H}_{k1} = kC_4\mu_{k1},$$

$$\lambda_{k1} = \frac{5}{2\alpha(\alpha + 2ks)}\{b_0B_0[3s(\alpha + ks) + kb_2(\alpha - 2ks + b_2)] -$$

$$- kb_0[2B_1s(s + k\alpha) + b_2(A_1\alpha + b_2B_1)]\},$$

$$\mu_{k1} = \frac{5}{2\alpha(\alpha + 2ks)}\{b_1B_0[3ks(b_3 - \alpha) + b_2(\alpha + ks) - 2s^2] -$$

$$- kb_0[B_1s(2b_3 + b_2) + k\alpha(A_1s - kb_2B_1)]\}, \quad (k = \pm 1);$$

$$G_{2k} = -(C_3^2 - C_4^2)\lambda_{2k},\ H_{2k} = 2C_3C_4\lambda_{2k},\ \tilde{G}_{2k} = 2C_3C_4\mu_{2k},$$

$$\lambda_{2k} = \frac{5}{8}b_1\frac{1}{s^2 - (2s + k\alpha)^2}\{s^2[6s + k(3\alpha + b_2)] - kb_2[\alpha b_2 - b_2^2 - 2k\alpha s]\},$$

$$\mu_{2k} = \frac{5}{8}b_1\frac{1}{s^2 - (2s + k\alpha)^2}\left\{3s^2\left(b_2 + b_3 - \alpha - \frac{4}{3}s\right) + kb_2(2\alpha s + 2b_2s + k\alpha b_2)\right\},$$

$$\tilde{H}_{2k} = \mu_{2k}(C_3^2 - C_4^2),\ G_{12} = C_4\lambda_{02},\ \tilde{H}_{12} = C_4\lambda_{12},\ (k = \pm 1);$$

$$H_{12} = C_3\mu_{02}, \quad \tilde{G}_{12} = C_3\mu_{12},$$

$$\lambda_{k2} = \frac{-5b_1}{16\alpha(\alpha + s)}[(2\alpha + s - b_{2+k})(A_1b_2 + 3B_1s) -$$

$$- (-1)^k(2\alpha + s + b_{2+k})(A_1s - b_2B_1)], \quad (k = 0, 1);$$

$$\mu_{k2} = \frac{5b_1}{16\alpha(\alpha + s)}[-A_1b_2(2\alpha + s + b_{2+k}) + (-1)^k(2\alpha + s - b_{2+k}) \times$$

$$\times (3B_1s - A_1s + b_2B_1)], \qquad (k = 0, 1);$$

$$G_{-12} = C_4\lambda_{03},\ \tilde{H}_{-12} = C_3\lambda_{13},\ H_{-12} = C_3\mu_{03},\ \tilde{G}_{-12} = C_3\mu_{13},$$

$$\lambda_{k3} = \frac{5b_1}{16\alpha(s - \alpha)}[(2\alpha - s - b_{2+k})(A_1b_2 - 3B_1s) -$$

$$- (-1)^k(2\alpha - s + b_{2+k})(A_1s + b_2B_1)],$$

$$\mu_{k3} = \frac{-5b_1}{16\alpha(s - \alpha)}(2\alpha - s - b_{2+k})(-A_1b_2 + 3B_1s - A_1s - b_2B_1),$$

$$G_{10} = \frac{5}{4}\frac{C_4}{s}\left[b_1(b_2 + s)\left(\frac{b_2B_1}{s} - A_1\right) + 4b_0sB_0 + b_1b_2B_1\right],$$

$$H_{10} = \frac{5}{4}C_3\left[b_1(A_1 + B_1)\left(\frac{b_2^2}{s^2} - 1\right) + 4b_0B_0 - 2b_1B_1\right],$$

$$f_1 = \frac{2b - \alpha + b_2}{s^2 - (\alpha - 2b)^2}, \qquad f_2 = \frac{2b - \alpha + b_3}{s^2 - (\alpha - 2b)^2},$$

$$A_{00} = \frac{5}{2}\frac{b_0}{b_2}\left[\frac{1}{2}A_1^2 + 2B_0^2 + B_1^2 + (C_3^2 + C_4^2)\left(\frac{1}{2}b_2^2 + s^2\right) -\right.$$

$$\left.- \frac{3}{5}(C_1^2 + C_2^2) - \frac{b_1}{b_0}B_0(3B_1 + A_1)\right],$$

$$B_2 = \frac{5}{4}C_4\left[4b_0b_3B_0 - (b_2 + s)b_1\left(B_1 + \frac{b_3}{s}A_1\right) - 2sb_1B_1\right],$$

$$B_3 = \frac{5}{4}C_3[4b_0b_3B_0 - (b_2 + b_3)b_1(A_1 + B_1) - 2b_1b_3B_1].$$

Turning now to an analysis of these equations, we first consider the structure of the mixed terms which arise due to the presence of the resonant frequencies. The next terms in the expression for h_2 and k_2 can be written in the form $e_2 = \varkappa\sqrt{C_1^2 + C_2^2}\,t$, where e_2 is the eccentricity of the satellite orbit in the second approximation. Accordingly, the eccentricity has a sinusoidal secular variation in the first approximation, while its behavior in the second approximation is governed by the sign of $\varkappa$. It should be noted that $\varkappa$ is a function of the major semiaxis a, but it is quite difficult to find an explicit expression for the polynomial a. Numerical analysis that the equation $\varkappa=0$ has two positive roots on the integral $0<a\leqslant 10^5$: $a_1=26\,632$ km and $a_2=26\,634$ km. On the interval $a_1<a<a_2$ the function $\varkappa(a)$ has a discontinuity of the second kind.

Table 1

ij	M_{ij}/α_{ij}	N_{ij}/α_{ij}	$\widetilde{M}_{ij}/\alpha_{ij}$	$\widetilde{N}_{ij}/\alpha_{ij}$
00	$2C_2\beta_{00}$	$C_2\varkappa$	0	0
01	$C_2(\beta_{01} - 7\alpha b_1 B_0)$	$C_1[\gamma_{01} + 7\alpha(b_0B_1 - b_1B_0)]$	$C_1(\gamma_{01} + \beta_{0-1})$	$C_2(\beta_{01} + \beta_{0-1})$
$0-1$	$C_2(\beta_{01} - 33\alpha b_1 B_0)$	$C_1[\gamma_{01} - \alpha(7b_0B_1 + 33b_1B_0)]$	$-C_1(\gamma_{01} - \gamma_{0-1})$	$C_2(\beta_{01} - \gamma_{0-1})$
10	$\overline{C}_{24}\beta_{10}$	$C_{14}\beta_{10}$	$-C_{14}\gamma_{10}$	$\overline{C}_{24}\gamma_{10}$
-10	$C_{24}\beta_{-10}$	$\overline{C}_{14}\beta_{-10}$	$-\overline{C}_{14}\gamma_{-10}$	$C_{24}\gamma_{-10}$
02	$C_2[\beta_{02} - \alpha(7B_1 + 13A_1 - 20A_2)]$	$C_1\gamma_{02}(7\alpha - 19b)$	$C_1\gamma_{02}(45\alpha + 19b)$	$C_2[\beta_{02} + 45\alpha(B_1 - A_1)]$
$0-2$	$C_2[\beta_{0-2} + \alpha(13A_1 - 33B_1 + 20A_2)]$	$C_1\gamma_{0-2}(19b - 33\alpha)$	$C_1\gamma_{0-2}(5\alpha - 19b)$	$C_2[\beta_{0-2} - 5\alpha(B_1 - A_1)]$
11	$\overline{C}_{24}\beta_{11}$	$C_{14}\beta_{11}$	$-C_{14}\gamma_{11}$	$\overline{C}_{21}\gamma_{11}$
$1-1$	$-\overline{C}_{24}\beta_{1-1}(b_2 - s)$	$C_{14}\left[-\gamma_{1-1} + \frac{1}{2}(\alpha - s)(33b_2 + 7s)\right]$	$C_{14}\left[\gamma_{1-1} + \frac{5}{2}(\alpha - s)(s - b_2)\right]$	$-\overline{C}_{24}[\beta_{1-1} - 14(s - \alpha)](b_2 - s)$
-11	$C_{24}\beta_{-11}$	$\overline{C}_{14}[\gamma_{-11} - 38(s - \alpha)]$	$-\overline{C}_{14}\gamma_{-11}$	$C_{24}\gamma_{-11}$
$-1-1$	$C_{24}\beta_{-1-1}(s + b_2)$	$\overline{C}_{14}[\gamma_{-1-1} + (s + \alpha)(7s - 33b_2)]$	$\overline{C}_{14}[\gamma_{-1-1} + 5(b_2 + s)(s + \alpha)]$	$C_{24}(s + b_2)[\beta_{-1-1} + 28(s + \alpha)]$

Table 2

ij	00	01	$0-1$	10	-10	02
α_{ij}	$\frac{1}{b}$	$-\frac{1}{\alpha(2b + \alpha)}$	$\frac{1}{\alpha(2b - \alpha)}$	$-\frac{b_0}{2b + s}$	$\frac{b_0}{2b - s}$	$\frac{-b_1}{4\alpha(\alpha + b)}$
β_{ij}	$3b_1(A_1 + B_1) - b_0B_0$	$b(5b_0B_1 + 38b_1B_0)$	$5\alpha(b_0B_1 + 9b_1B_0)$	$12b + 7s$	$12b - 7s$	$b(19B_1 - 29A_1 + 10A_2)$
γ_{ij}	0	$2b(6b_0B_0 + 19b_1B_0)$	$5\alpha(b_0B_1 + b_1B_0)$	$12b + 5s$	$12b - 5s$	$A_1 - B_1$

ij	$0-2$	11	$1-1$	-11	$-1-1$
α_{ij}	$\frac{b_1}{4\alpha(b - \alpha)}$	$\frac{-b_1(b_2 + s)}{(s + \alpha)(2b + s + \alpha)}$	$\frac{-b_1}{(s - \alpha)(2b + s - \alpha)}$	$\frac{b_1(s - b_2)}{2(s - \alpha)(2b + \alpha - s)}$	$\frac{b_1}{2(s + \alpha)(2b - s - \alpha)}$
β_{ij}	$b(19B_1 - 9A_1 - 10A_2)$	$19b - \frac{7}{2}(s + \alpha)$	$19b + \frac{33}{2}(s - \alpha)$	$38b + 7(s - \alpha)$	$38b - 33(s + \alpha)$
γ_{ij}	$-A_1 + B_1$	$19b + \frac{45}{2}(s + \alpha)$	$b(19b_2 + s)$	$38b + 45(\alpha - s)$	$2b(19b_2 - s)$

This behavior is found near the resonant major semiaxis $a^*=26\,631$ km, indicated in [1]. On the interval $a_2<a<a_1$ the function $\varkappa(a)<0$ and thus the orbital eccentricity e_2 increase if $a<a_1$, or decrease if $a>a_2$. The next terms in the expressions for p_2 and q_2 are

$$\sin\frac{i}{2}\sqrt{\frac{b_3}{b_2}\sin^2\Omega+\cos^2\Omega}=t\sqrt{B_2^2+B_3^2}. \tag{7}$$

With $a=6\cdot10^4$ km we have b_3/b_2 = 1.061, while with $a=10^5$ km we have b_3/b_2 = 1.006, so on the interval $6\cdot10^4<a<10^5$ we can use the approximation b_2 = b_3; then Eq. (7) becomes

$$\sin\frac{i}{2}=\frac{5}{2}b_2t(2b_0B_0-2b_1B_1-b_1A_1)\sqrt{C_3^2+C_4^2}.$$

Calculations show that $2b_0B_0-2b_1B_1-b_1A_1$ vanishes at $a=80\,400$ km, and that if we have $6\cdot10^4<a<80\,400$, we have $\sin\frac{i}{2}<0$, and if we have $8{,}04\cdot10^4<a<10^5$, we have $\sin\frac{i}{2}>0$. If the major semiaxis of the orbit satisfies $a<6\cdot10^4$ km, the secular changes in the orbital information must be treated jointly with the secular changes in the longitude of the ascending node, in accordance with (7).

In conclusion we emphasize, despite the presence of the next terms, the coefficients of t in these terms are never greater 10^{-7}, so the equations derived can be used for quite long time intervals. We will postpone the analysis of the periodic terms in the secular perturbations, merely noting that the equations have no singularities other than those mentioned in [1].

REFERENCES

1. V. P. Dolgachev, Vestn. Mosk. un-ta, fiz., astron. [Moscow University Physics Bulletin], No. 5, 1969.
2. V. P. Dolgachev, Soobshcheniya GAISh, Izd-vo MGU, No. 154, 1968.
3. G. N. Duboshin, Basic Theory of the Stability of Motion [in Russian], Izd-vo MGU, 1952.

28 April 1969

Department of Stellar Mechanics and Gravimetry