RADIATION POLARIZATION OF SLIGHTLY RELATIVISTIC ELECTRONS IN A MAGNETIC FIELD

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In an analysis of the effect of external electromagnetic radiation on the spin state of slightly relativistic electrons in a uniform magnetic field, it is shown that an electron beam can become highly polarized under certain conditions. An expression is found for the probability for radiative transitions involving spin flipping which holds for all particle energies.

INTRODUCTION

Self-polarization was first demonstrated for a system of alterrelativistic electrons moving along circular orbits in a constant and uniform magnetic field [1-3]. It arises because some of the photons emitted during the circular motion result from quantum transitions involving a change in the spin orientation. The total probability for transitions from a state with a spin parallel to the field to a state with the spin anti-parallel to the field $w^{\uparrow\downarrow}$, is much greater than the probability $w^{\downarrow\uparrow}$ for the opposite transitions. As a result of the a symmetry in these spin-flipping transitions, and initially unpolarized electron beam will become 96% polarized opposite the magnetic field after a time on the order of 1 hr ($E \sim 1$ GeV, $H \sim 10^4$ G).

Nonrelativistic spin-flipping amplitudes also display this asymmetry property with respect to transfer spin. The radiation polarization of a nonrelativistic electron beam in a magnetic field was analyzed in [4] (where there is an error in the numerical value of the coefficient in the expressions for the transition probabilities). The polarization time turns out to be six orders of magnitude greater in the nonrelativistic than in the ultrarelativistic case, so the effect would seem to be of little practical interest. However, the situation changes radically when there is an external resonant electromagnetic wave which stimulates radiative transitions. In this case transitions are possible involving both the emission and absorption of photons from the external wave. It follows from the principle of detailed equilibrium that the probability for a $\uparrow\downarrow$ transition involving emission of the photon is equal to that for a $\downarrow\uparrow$ transition involving absorption of a photon. It must be taken into account here, however, that the initial energies differ by a small amount, and if we assume δ-shaped momentum distribution in the beam we can tune the frequency of the external wave with respect to the transition frequency to cause emission to prevail above absorption or vice-versa. This is actually the principle of which the operation of so-called cyclotron-resonance masers is based [7]. It is not difficult to see that during operation in the "maser" mode, the net probability for induced spin-flipping transitions will display the asymmetry necessary for the appearance of polarization. We would expect the induced radiation polarization to be important in the case of slightly relativistic electrons. The total probability for a polarization change in the ultrarelativistic case consists of a large number of terms corresponding to various harmonics; this number is much greater than the six orders of magnitude by which the total probabilities in the nonrelativistic and ultrarelativistic cases differ. Since the intensity in the incident wave must have a quite sharp spectral distribution in order for emission to prevail over absorption (a single harmonic must be singled out), and since the total probability in the nonrelativistic case also contains only a single term, corresponding to the first harmonic, we see that the effect of induced transitions should be important in precisely this case.

§1. SPONTANEOUS TRANSITIONS

Before turning to a detailed analysis of the electron polarization in the presence of an external wave, we will correct the expression found in [4] for the probability of spontaneous spin-flipping transitions in the nonrelativistic approximation. In contrast with [4], we start from the nonrelativistic equations with an account of the spin corrections ($\sim v/c$):

$$\left(-\frac{\hbar}{i}\frac{\partial}{\partial t}-\mathscr{H}\right)\Psi(N+1)=U^{+}\Psi(N),$$
$$\left(-\frac{\hbar}{i}\frac{\partial}{\partial t}-\mathscr{H}\right)\Psi(N)=U\Psi(N+1), \tag{1}$$

where N is the number of photons, $\mathscr{H}$ is the Pauli Hamiltonian (see, e.g., [8]),

$$U=-\frac{i}{L^{3/2}}\frac{e\hbar}{2mc}\sum_{\varkappa,\lambda}\sqrt{\frac{2\pi c\hbar}{\varkappa}}\,\vec{\sigma}\,[\vec{e}_{\lambda},\vec{\varkappa}]\,e^{-ic\varkappa t}a_{\varkappa,\lambda}, \tag{2}$$

$$U^{+} = \frac{i}{L^{3/2}} \frac{e\hbar}{2mc} \sum_{\varkappa,\lambda} \sqrt{\frac{2\pi c\hbar}{\varkappa}}\, \vec{\sigma}\,[\vec{e}_{\lambda}, \vec{\varkappa}]\, e^{ic\varkappa t} a^{+}_{\varkappa,\lambda}, \tag{2}$$

are the operators representing the interaction with the second-quantized photon field in the magnetic-dipole approximation, and $a_{\varkappa,\lambda}$, $a^{+}_{\varkappa,\lambda}$ are the annihilation and creation operators [8] for photons having momentum $\vec{\varkappa}$ and polarization parallel to the unit vector $\vec{e}_{\lambda}$.

The eigenfunctions and corresponding eigenvalues of the Hamiltonian are well-known and are given, e.g., [9].

In interaction (2) we have obtained only the terms responsible for emission involving a spin-flipping. The transitions of interest here, at the cyclotron frequency and involving a change in the spin projection on the $\vec{H}$ direction, should occur been states $(n, \zeta=1)$, $(n, \zeta=-1)$ or $(n, \zeta=-1)$, $(n-2, \zeta=1)$. Assuming a vanishing momentum p_z parallel to the field (for comparison with [4]), we find desired probability for spin-flipping transitions as a function of the initial spin orientation $\zeta=\pm 1$ to be

$$w = \frac{2}{3} \frac{1}{T_0} \left(\frac{H}{H_0}\right)^3 \frac{1+\zeta}{2}, \quad T_0 = \frac{\hbar^2}{mce^2}, \quad H_0 = \frac{m^2c^3}{e\hbar}. \tag{3}$$

The numerical value of the coefficient in Eq. (3) differs from that in [4]. We see that magnetic-dipole transitions involving a change in the spin projection on the z axis occur only from the state $\zeta=1$ (with the spin parallel to the field). Accordingly, over a relaxation time $\sim \frac{1}{w}$ the beam becomes completely polarized opposite the field ($\frac{1}{w} \sim 10^{10}$ sec).

We can also find a general expression for the probability of transitions involving spin flipping which is valid in both the nonrelativistic and ultrarelativistic cases. For this purpose, we note that [2] that the probability for the emission of hard photons decreases very rapidly with increasing value of ratio $\frac{\hbar\omega}{E}$; the only contribution comes from quite soft photons, for which $\frac{\hbar\omega}{E} \ll 1$. This is to be expected in view of the semi-classical nature of the motion. Using a method analogous to that derived in [2], and exploiting the smallness of the ratio $\frac{\hbar\omega}{E}$, we find an expression for the probability of spontaneous spin-flipping transitions which is applicable over the entire electron energy range:

$$\begin{aligned} w_{sp} = \frac{1}{2T_0} \left(\frac{H}{H_0}\right)^3 \frac{\varepsilon_0^{5/2}}{\beta} \sum_{\nu=1}^{\infty} \nu^2 \Big\{ J'_{2\nu}(2\nu\beta) + \frac{\varepsilon_0}{2\nu\beta} J_{2\nu}(2\nu\beta) + \\ + \frac{\varepsilon_0}{4}\left(1 - \frac{1}{\nu^2\beta^2}\right) \int_0^{2\nu\beta} J_{2\nu}(x)\,dx - \frac{\varepsilon_0}{16\nu^2\beta^2} \int_0^{2\nu\beta} x^2 J_{2\nu}(x)\,dx + \\ + \zeta \frac{\sqrt{\varepsilon_0}}{\beta} \left[J_{2\nu}(2\nu\beta) + \frac{1}{2\nu\beta} \int_0^{2\nu\beta} J_{2\nu}(x)\,dx \right] \Big\}. \end{aligned} \tag{4}$$

The summation over ν corresponds to the summation over the various frequency harmonics $\omega_c = \frac{eHc}{E}$.

For slightly relativistic energies($\nu=1$, $\beta \ll 1$)Eq. (4) yields, with an accuracy to within small terms

$$w_1 = \frac{2}{3} \frac{1}{T_0} \left(\frac{H}{H_0}\right)^3 \left[(1 - 3{,}3\beta^2) \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \beta^4 a \right], \tag{5}$$

where $a<1$ is a numerical coefficient.

In the ultrarelativistic case, the summation can be replaced by an integration, with the Bessel functions approximated by Macdonald functions [8] (the expansion primary is $\varepsilon_0 = 1-\beta^2$):

$$w_2 = \frac{5\sqrt{3}}{36} \frac{e^2}{R\hbar} \frac{E}{mc^2} \xi^2 \left(1 + \zeta \frac{8\sqrt{3}}{15}\right), \quad \xi = \frac{3}{2\sqrt{\varepsilon_0}} \frac{H}{H_0}, \quad R \simeq \frac{E}{eH}. \tag{6}$$

Equation (6) agrees exactly with that derived in [1, 2] through expansions only in terms of the parameter ε_0 (also with a formal account of the terms $\hbar\omega \sim E$). Accordingly, hard photons make no contribution to the probability in this approximation.

§2. INDUCED ELECTRON POLARIZATION

We turn now to the effect on the electron polarization of induced transitions which occur in the presence of an external wave at a frequency near the cyclotron frequency. As was mentioned above, an asymmetry in the spin transitions with respect to the initial spin direction arises only when transitions involving emission and absorption are not equiprobable. We can achieve such a situation if the lines in energy spectrum differ even slightly $\left(\sim \frac{\hbar\omega}{E}\right)$ from an equidistant arrangement. Since the energy levels generally have a finite width, $\sim\frac{\hbar}{\tau}$, where τ is the lifetime, a wave whose frequency is within $\sim\frac{1}{\tau}$ of the cyclotron frequency will also induce quantum transitions, with a probability which depends on the frequency difference. Taking into account the finite lifetime, we should replace the δ-function expressing energy conservation by a Lorentz factor:

$$\delta(\omega-\omega_{nn'})\rightarrow\frac{1}{2\pi}g_{nn'}(\omega)=\frac{1}{\pi}\frac{\tau}{1+x_{nn'}^2},\quad x_{nn'}=(|\omega_{nn'}|-\omega)\tau.$$

We see that if $\omega_{n,n-\nu}\neq\omega_{n+\nu,n}$, we can choose a frequency difference between the external frequency and the resonant frequency (i.e., we can choose the quantity x) such that transitions involving either emission or absorption will prevail.

For numerical estimates, we rate the probability of induced spin-flipping transitions in terms of the electric field $\mathcal{E}_\lambda$ in the wave polarized parallel to $\vec{e}_\lambda$:

$$w_{nn'}^\lambda=\frac{e^2c^2}{2\hbar^2\omega^2}\mathcal{E}_\lambda^2\Phi_\lambda g_{nn'}(\omega). \tag{7}$$

With the standard choice of $\vec{e}_\sigma=(-1,0,0)$, $\vec{e}_\pi=(0,\cos\theta,-\sin\theta)$ as $\vec{e}_\lambda$, and assuming $p_z=0$, we can write [2]

$$\begin{gathered}\Phi_\sigma=\bar{\alpha}_1\bar{\alpha}_1^*,\quad \Phi_\pi=|\bar{\alpha}_2\cos\theta-\bar{\alpha}_3\sin\theta|^2;\\ \bar{\alpha}_1=-i\frac{z}{4n}\beta\operatorname{ctg}\theta\left[\frac{\nu}{z}J_\nu(z)+\zeta\sqrt{\varepsilon_0}J'_\nu(z)\right],\\ \bar{\alpha}_2=\frac{z}{4n}\beta\operatorname{ctg}\theta\left[J'_\nu(z)+\zeta\sqrt{\varepsilon_0}\frac{\nu}{z}J_\nu(z)\right],\\ \bar{\alpha}_3=-\frac{z}{4n}\beta\left[J'_\nu(z)+\zeta\sqrt{\varepsilon_0}\frac{\nu}{z}J_\nu(z)\right],\\ z=\frac{\omega}{\omega_c}\beta\sin\theta,\end{gathered} \tag{8}$$

where $\nu=n-n'$ when $n>n'$, Eq. (7) predicts transitions involving emission; $n<n'$, it predicts transitions involving absorption. In Eq. (7) we have neglected the difference between the frequency for the $n\rightarrow n+|\nu|$ and $n\rightarrow n-|\nu|$ except in the factor $g_{nn'}$. Restricting our attention now to the slightly relativistic case ($n'=n\pm1$, $\beta\ll1$),, we find the total probability for spin-flipping to be

$$w^\lambda(\zeta)=\frac{e^2\mathcal{E}_\lambda^2\tau}{8m^2c^2}G_\lambda^2\left[\frac{1+\zeta}{1+x^2}+\frac{1-\zeta}{1+(x-2q)^2}\right], \tag{9}$$

where $x=\tau(\Omega-\omega)$,

$$\begin{gathered}G_\sigma=\cos\theta,\quad G_\pi=1;\\ q=\tau\Omega\frac{H}{2H_0}\left(1-\frac{\omega^2}{\Omega^2}\cos^2\theta\right),\end{gathered} \tag{10}$$

and $\Omega=\frac{eH}{mc}$ is the nonrelativistic cyclotron frequency. The parameter q characterizes the deviation of the level arrangement from an equidistant arrangement which results from the relativistic corrections:

$$q=\frac{\tau}{2}(\omega_{n,n-1}-\omega_{n+1,n}). \tag{11}$$

For probability (9) to depend significantly on ζ, we must evidently have $q\sim1$; an angle of incidence $\theta=\frac{\pi}{2}$ is thus the most favorable (i.e., with wave vector in the orbital plane).

With $H\sim10^4$ G the lifetime τ (the effective interaction time) must be no shorter than 10^{-3}—10^{-2} sec; this restriction immediately imposes quite severe limitations on the electron velocity.

In real situations, τ may be governed by spontaneous emission, transit time across the interaction region, collisions with residual gas modules, etc. The effects of all factors except the first can in principle be eliminated; spontaneous emission gives rise to the so-called natural level width, which is evidently the minimum width possible. In our case the effective time for the spontaneous emission is [10]

$$\tau=\frac{3}{2}\frac{\hbar c}{e^2}\frac{1}{\Omega\beta^2}\simeq\frac{2\cdot10^3}{\Omega\beta^2}\ (\text{sec}). \tag{12}$$

Comparing Eqs. (12) and (10), we find that the asymmetry of (9) with respect to ζ arises only for velocities below $10c\sqrt{\frac{H}{H_0}}$. (This is true only in the slightly relativistic case, for which dipole transitions are primarily imporant.)

We define the effective time and polarization coefficient by

$$T_p=\frac{1}{\sum\limits_\zeta w^\lambda(\zeta)},\quad k=\lim_{t\rightarrow\infty}\frac{n^\downarrow(t)}{n^\uparrow(t)+n^\downarrow(t)}=\frac{w^\lambda(\zeta=1)}{\sum\limits_\zeta w^\lambda(\zeta)} \tag{13}$$

($n^{\uparrow(\downarrow)}(t)$ is the number of electrons polarized parallel to (or antiparallel to) the field). According to Eqs. (10), (12) and (13), we have

$$T_p = 2\cdot 10^{-2}\left(\frac{mc\beta}{e\mathcal{E}_{\lambda}G_{\lambda}}\right)^2 \Omega k(1+x^2);\ k=\frac{1}{2}\,\frac{1+(x-2q)^2}{1+q^2+(x-q)^2}. \tag{14}$$

For unpolarized external radiation, k is unchanged, but the effective polarization time becomes

$$T_p = 2\cdot 10^{-2}\left(\frac{mc\beta}{e\mathcal{E}}\right)^2 \Omega\,\frac{k(1+x^2)}{1+\cos^2\theta}. \tag{15}$$

As was noted above, the wave vector must lie in the orbital plane. From Eqs. (14) and (15) we see that in this case the wave polarization is relatively unimportant.

The polarization coefficient and time are functions of x which represents the deviation of the external frequency from the resonant frequency of the transition. It is easy to see, in general, that by choosing various x values we can achieve a beam polarization parallel to or antiparallel to the field. In practice, however, the region $k<\frac{1}{2}$ is of little interest (with polarization parallel to the field), since the corresponding x values are large. The maximum value of k(x) is found at $x(q) = q-\sqrt{q^2+1}$;

$$k(q)=\frac{1}{2}\left(1+\frac{q}{\sqrt{1+q^2}}\right). \tag{16}$$

In this case in which we are assuming the natural level width, we have

$$q=\frac{10^2}{\beta^2}\,\frac{H}{H_0}. \tag{17}$$

Finally, the polarization time for $x=x(q)$ and $\theta=\frac{\pi}{2}$ is $T_p(q)=2\cdot 10^{-2}\left(\frac{mc\beta}{e\mathcal{E}}\right)^2\Omega$.

With $\beta\sim 0.15\cdot 10^{-3}$, $H\sim 10^4$ G we thus have $k\sim 0{,}86$ and an effective time of $T_p\sim\frac{2{,}6\cdot 10^{-13}}{\mathcal{E}^2}$ sec (the field intensity is given in oersteds).

$$\omega_{n,n\pm 1}=\Omega\mp\zeta\frac{\alpha\mu_0}{\pi c\hbar},\quad \mu_0=\frac{e\hbar}{2mc},\quad \alpha=\frac{e^2}{c\hbar}.$$

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