PETROV CLASSIFICATION OF GRAVITATIONAL FIELDS AND CHRONOMETRIC INVARIANTS

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Vestnik Moskovskogo Universiteta. Fizika, Vol. 25, No. 3, pp. 350-352, 1970

UDC 530.12: 531.50

The reference frame is defined in some region of V_4 as the congruence of time lines [1] to which the unit time-life vector $u^\alpha = dx^\alpha/ds$ are tangent. The Weyl tensor $C_{\mu\alpha\nu\beta}$ splits into two tensors each having five important components:

$$U_{\mu\nu} = C_{\mu\alpha\nu\beta}u^\alpha u^\beta, \quad V_{\mu\nu} = {}^*C_{\mu\alpha\nu\beta}u^\alpha u^\beta,$$

$${}^*C_{\mu\alpha\nu\beta} \equiv \frac{1}{2}\varepsilon_{\mu\alpha\rho\sigma}C^{\rho\sigma}_{\nu\beta}, \quad U_{[\mu\nu]} \equiv \frac{1}{2}(U_{\mu\nu} - U_{\nu\mu}) = 0,$$

$$V_{[\mu\nu]} = 0, \quad \mathrm{Sp}\, U = \mathrm{Sp}\, V = 0$$

(here the Greek indices run over 1, 2, 3, 0; and the Latin indices run over 1, 2, 3). This splitting is obviously invariant with respect to internal coordinate transformations in the given reference frame: $\tilde{x}^\alpha = \tilde{x}^\alpha(x^\beta)$, $\partial\tilde{x}^i/\partial x^0 = 0$.

Zel'man has shown that the Riemann tensor splits into three chronometrically invariant tensors [2]. In a semi-geodesic reference frame (the corresponding equations are denoted by asterisks), these tensors are

$$x_{ij} \overset{*}{=} \partial D_{ij}/\partial t - D_{il}D^l_j, \quad Y_{kij} \overset{*}{=} \nabla_j D_{ik} - \nabla_k D_{ij},$$

where $Z_{iklj} \overset{*}{=} D_{ik}D_{lj} - D_{il}D_{kj} - C_{iklj}$, $2D_{ik} \overset{*}{=} -\partial g_{ik}/\partial t$; D_{ik}, C_{iklj}, and D_{ik} and C_{iklj} are the chronometrically invariant deformation-rate tensor and the curvature tensor for the given space of the reference frame.

The four-dimensional representations of the tensors X_{ik} and Y_{ik} of the dual Y_{kij} and of the tensor Z_{ik} of the double dual Z_{iklj} are [3]

$$X_{\mu\nu} = R_{\mu\alpha\nu\beta}u^\alpha u^\beta, \quad Y_{\mu\nu} = {}^*R^*_{\mu\alpha\nu\beta}u^\alpha u^\beta, \quad Z_{\mu\nu} = {}^*R^*_{\mu\alpha\nu\beta}u^\alpha u^\beta.$$

In vacuum, we have $U_{\mu\nu} = X_{\mu\nu} = -Z_{\mu\nu}$ and $V_{\mu\nu} = Y_{\mu\nu}$. The tensor $U_{\mu\nu}$ is governed by a 3 x 3 matrix (U_{ik}). In matrix form, we have $(C_{\mu\alpha\nu\beta}) = \begin{pmatrix} U & iV \\ iV & U \end{pmatrix}$. We introduce the coordinate $x^0 = it$ and write the tensor of even(fourth) valence as a matrix (6 x 6). Its characteristic equation is [6]

$$-|C_{\mu\alpha\nu\beta} - \lambda g_{\mu\alpha\nu\beta}| = \lambda^3 - 3I_1\lambda - 2I_2 = 0,$$

$$g_{\mu\alpha\nu\beta} \equiv g_{\mu\nu}g_{\alpha\beta} - g_{\mu\beta}g_{\alpha\nu}, \quad I_1 = \frac{1}{48}(C_{\mu\alpha\nu\beta}C^{\mu\alpha\nu\beta} - iC_{\mu\alpha\nu\beta}{}^*C^{\mu\alpha\nu\beta}),$$

$$I_2 = \frac{1}{96}(C_{\mu\alpha\nu\beta}C^{\nu\beta\rho\sigma}C^{\mu\alpha}_{\rho\sigma} + iC_{\mu\alpha\nu\beta}C^{\nu\beta\rho\sigma}{}^*C^{\mu\alpha}_{\rho\sigma}).$$

This is evidently the same equation as that for the complex symmetric matrix A = U + Vi. We find three complex roots

$$\lambda_{1,2,3} = \varepsilon_{1,2,3}\sqrt[3]{I_2 + \sqrt{d}} + \varepsilon_1 \pm i\sqrt{3/2}\sqrt[3]{I_2 - \sqrt{d}},$$

$$d = I_2^2 - I_1^3, \quad \varepsilon_1 = 1, \quad \varepsilon_{2,3} = -1/2 \pm i\sqrt{3/2},$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 0, \quad \lambda_i = \alpha_i + i\beta_i.$$

From the Hamilton-Cayley theorem, we have $A^3 - 3I_1\lambda - 2I_2E = \prod_i (A - \lambda_i E) = 0$. The solutions of this equation can be grouped in three classes: $a\,(d \neq 0)$, $b\,(d = 0,\ I_1 \neq 0)$, and $c\,(I_1 = I_2 = 0)$. For class a we can find $A = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. For class b, we have $(A - \lambda_1 E)(A - \lambda_2 E)^2 = 0$. The complex Euclidean space E_3^+ splits into two invariant subspaces, and A is quasidiagonal in the appropriate basis. The 1 x 1 cell is of the form λ_1, while for a 2 x 2 we find $A - \lambda_2 E = 0$ и $A - \lambda_2 E = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \equiv A_N \neq 0$, $A_N^2 = 0$; then we find $A_N = \alpha\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ where α is any complex number. We assume α = 1. For class c, we have $A^3 = 0$; i.e., three matrices correspond to the three nilpotent indices: the null matrix A_N augmented by null matrices (which we denote as $A_{II\,d}$), and $A_{III} = \begin{pmatrix} \cdot & 1 & i \\ 1 & \cdot & \cdot \\ i & \cdot & \cdot \end{pmatrix}$. We find the latter matrix by using $A_{III}^2 = A_{II\,d}$. The relationship between these classes and the Petrov classes is obvious.

The matrices U and V are canonical in some basis and E_3^+. This basis determines in V_4 a canonical tetrad with an unambiguously defined time-like vector corresponding to some reference frame. The reference frame in which the splitting tensors can be simultaneously reduced to canonical form by internal coordinate transformations is the "canonical" reference frame.

From the condition $A^2=(U+iV)^2=0$ for type IId we find that U and V anticommute. For spaces T_3(type III), we have $UVU=VUV=0$ while for T_1 we have UV = VU. [For type II $(UV-VU)^2$ is idempotent.] Then we find the tensor relations to be

$$C_{\rho\beta\alpha[\mu}{}^*C_{\nu]\gamma\delta\sigma}\, g^{\rho\sigma}u^{\alpha}u^{\beta}u^{\gamma}u^{\delta}=0 \quad (T_1),$$

$$C_{\rho\beta\alpha(\mu}{}^*C_{\nu)\gamma\delta\sigma}\, g^{\rho\sigma}u^{\alpha}u^{\beta}u^{\gamma}u^{\delta}=0 \quad (\mathrm{II}\,d),$$

$$C_{\mu\alpha\sigma\beta}{}^*C_{\lambda\xi\tau\eta}\, C_{\nu\gamma\rho\delta}\, g^{\lambda\sigma}g^{\tau\rho}u^{\alpha}u^{\beta}u^{\gamma}u^{\delta}u^{\xi}u^{\eta}=0 \quad (\mathrm{III})\ (A).$$

We can assert, e.g., that a necessary and sufficient condition for the Weyl tensor in the region to be a type III tensor is the presence in it of a reference in which (A) halts, etc. (Type II is obtained more simply by elimination, while type I and Id can be distinguished on the basis of the discriminant d.) On the other hand, if the Weyl tensor in this region belongs to type III, Eq. (A) gives a congruence of time lines, (in other words, a canonical reference frame) with a given tangent vector u^{α} (with an accuracy within a single Lorentz rotation with the rotation parameter σ which appears in U and V).

For the limiting transition $u^{\alpha}\to l^{\alpha}$ (where ℓ^{α} is the isotropic Riemann principal vector), we find $\varphi\to i\infty$ and $\sigma^{r\cdot}$ for the corresponding complex angle φ; i.e., the splitting by means of ℓ^{α} for spaces T_2 and T_3 is same as that found with the help of the canonical vector u^{α} for spaces T_1 (the tensor relations also simplify). We call the corresponding congruence of isotropic lines a "degenerate canonical reference frame."

We consider a linear covariant approximation for the gravitational field in vaccum near a given world point P in a locally Galilean canonical frame, which is semigeodesic in the region. Then near P or even near coordinate line x', we have, within vanishingly small higher-order terms,

$$g_{0\alpha}=\delta_{0\alpha},\ -g_{ik}=\delta_{ik}+2\int D_{ik}dt,\ D_{ik}(P)=0,$$

$$\nabla_m=\partial/\partial x^m,\ A=\dot{D}+D^2+i\omega D=\dot{D}+i\omega D,\ \omega=(\varepsilon^{ijk}\nabla_k).$$

Using the canonical form of the matrix A, and assuming $D_{ij,k}(P)=0$ for simplicity for the null cells in A (the notation here is trivial), we find a local classification in terms of the deformation-rate tensor of this reference frame:

$$D_{\mathrm{I}}=\begin{pmatrix} \alpha_1 t & az & (a-\beta_1)\,y \\ az & \alpha_2 t & (a+\beta_1)\,y \\ (a-\beta_1)\,y & (a+\beta_1)\,y & \alpha_3 t \end{pmatrix},\quad a=D_{12,3}\ (P),$$

$$D_{\mathrm{I}d}=diag\,(0,\ t-x,\ -t+x)\ \ \text{for}\ \ D_{12,2}=D_{13,3}-0,$$

$$D_{\mathrm{II}d}=\begin{pmatrix} . & y & -z \\ y & t & . \\ -z & . & -t \end{pmatrix}\ \text{for}\ \ D_{22,1}=D_{32,1}=0,$$

$$D_{\mathrm{III}}=\begin{pmatrix} . & t-x & . \\ t-x & . & z \\ . & z & . \end{pmatrix}\ \text{for}\ \ D_{11,2}=-D_{33,2}=0,$$

$$D_{\mathrm{III}}=\begin{pmatrix} y & t & . \\ t & . & . \\ . & . & -y \end{pmatrix}\ \text{for}\ \ D_{12,1}=D_{32,3}=0,$$

$$D_{\mathrm{II}}=D_{\mathrm{I}d}+D_{\mathrm{II}d},\quad (x^0,\ x^1,\ x^2,\ x^3=t,\ x,\ y,\ z).$$

For a Schwarzschild field, e.g., we have $D=diag\left(\frac{2mt}{r^3},-\frac{mt}{r^3},-\frac{mt}{r^3}\right)$; i.e., there is extension along directions toward the central body and from it, and there is a half-symmetric compression in the orthogonal plane. For type IId, in particular, we find a locally planar deformation wave and three-dimensional curvature (X = −Z); for type III we find a variable-shear transverse longitudinal wave in the xy plane propagating parallel to the OX axis. In the other coordinate planes, the shear is parallel to Oy, constant at the wave front, and in correspondence with some rotation.

In conclusion the author thanks A. L. Zel'manov and N. V. Mitskevich for interest in this study.

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5 November 1969

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