CHRONOMETRICALLY INVARIANT FORMULATION OF THE SECOND LAW OF RELATIVISTIC THERMODYNAMICS

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Vestnik Moskovskogo Universiteta. Fizika, Vol. 25, No. 3, pp. 352-254, 1970

In the general theory of relativity, the first law of thermodynamics is the law expressing conservation of energy-momentum [1]:

$$T^{\mu\nu};_{\nu} = 0, \tag{1}$$

where $T^{\mu\nu}$ is the energy-momentum tensor.

The second law of thermodynamics can be written [1]

$$S^{\mu};_{\mu}\delta W \geqslant \frac{\delta Q}{T}, \tag{2}$$

where

$$S^{\mu} = \varphi dx^{\mu}/ds \tag{3}$$

is by definition the four-density of the entropy flux, φ is the proper entropy density, dx^{μ}/ds is the four-velocity of the element of the medium under consideration, δW is 4-volume corresponding to the 3-volume of the elements of the medium in the elementary time interval, δQ is the heat acquired by the element of the medium across the boundary of the volume it occupies during the elementary time interval, and T is the temperature of this boundary.

The general expression for the 4-volume element δW in terms of the vectors $\delta x^{\nu}_{(0)}$, $\delta x^{\nu}_{(1)}$, $\delta x^{\nu}_{(2)}$ and $\delta x^{\nu}_{(3)}$ on the basis of which it can be constructed is (here we are not choosing these vectors in a special manner described in [1])

$$\delta W = \sqrt{-g}\,\left|\,\mathrm{Det}\,|\,\delta x^{\nu}_{(\mu)}\,|\,\right|, \tag{4}$$

where $g = \mathrm{Det}\,|\,g_{\mu\nu}\,|$, $|g_{\mu\nu}|$ is the 4-metric tensor.

Because of their general covariance, Eqs. (1) and (2) are the most general equations, in the sense that they hold in any reference frames and for any gravitational field. In addition, because of this general covariance (4-dimensional), the difference between 3-space and the time is not reflected in them. However, this difference is important in all specific applications of the relativistic equation. Use of the formalism of chronometric invariants [2] permits space and time to be distinguished in these equations without restricting their generality.

The chronometrically invariant formulation of law (1) is known [2]; here we find the chronometrically invariant formulation of law (2).

We first transform the expression for δW. We denote by $\delta\tau_{\mu}$ the chronometrically invariant time interval which represents the temporal projection of the vector $\delta x^{\nu}_{(\mu)}$:

$$c\delta\tau_{(\mu)} = \frac{g_{0k}dx^{\alpha}_{(\mu)}}{\sqrt{g_{00}}} \tag{5}$$

where c is the speed of light. Here the Greek indices run over the values 0, 1, 2, 3; while the Latin indices run over the values 1, 2, 3.

We choose the vectors $\delta x^{\nu}_{(i)}$ to be spatial $(\delta\tau_{(i)} = 0)$, and choose the vector $\delta x^{\nu}_{(0)}$ to be temporal $(\delta x^{i}_{(0)} = 0)$. . (Incidentally, one of these conditions is sufficient) then, emitting the index (the zero) on $\delta\tau_{(0)}$, we can write

$$\delta W = c\sqrt{h}\;\delta\tau\left|\,\mathrm{Det}\,|\,\delta x^{i}_{(j)}\,|\,\right|, \tag{6}$$

where $h = \mathrm{Det}\,|\,h_{ij}\,|$, where h_{ij} is the chronometrically invariant 3-metric tensor [2]. For the chronometrically invariant 3-elementary volume we have

$$\delta V = \sqrt{h}\left|\,\mathrm{Det}\,|\,\delta x^{i}_{(j)}\,|\,\right|. \tag{7}$$

So we find

$$\delta W = c\delta\tau\delta V. \tag{8}$$

For the 4-divergence of any vector B^{μ} we have

$$B^{\mu};_{\mu} = \frac{1}{c}\left(\frac{{}^{*}\partial}{\partial t} + D\right)\frac{B_0}{\sqrt{g_{00}}} + \left({}^{*}\nabla - \frac{F_i}{c^2}\right)B^i. \tag{9}$$

Here B_0 and B^i are the covariant temporal and contravariant spatial components of the vector B^μ. The quantities $B_0/\sqrt{g_{00}}$ and B^i are a chronometrically invariant 3-scalar and a chronometrically invariant 3-vector, respectively. The chronometrically invariant scalar D is the trace of the deformation-rate tensor of the reference space, while F_i is the gravitational-inertial force permanent mass. Here and below, the asterisk denotes chronometrically invariant differentiation operators.

We turn now to Eq. (3); since

$$c\frac{d\tau}{ds} = \left(1 - \frac{h_{ij}v^i v^j}{c^2}\right)^{-1/2},$$

where

$$cd\tau = \frac{g_{0\alpha}dx^\alpha}{\sqrt{g_{00}}},$$

we have

$$\frac{S_0}{\sqrt{g_{00}}} = \frac{\varphi}{\sqrt{1 - \frac{h_{ij}v^i v^j}{c^2}}}, \quad S^i = \frac{\varphi v^i}{c\sqrt{1 - \frac{h_{ij}v^i v^j}{c^2}}}. \tag{10}$$

where S_0 and S^i are the covariant temporal and contravariant spatial components of the vector S^μ, and $v^i = dx^i/d\tau$ is the chronometrically invariant 3-velocity of the element of the medium. Evidently, $S_0/\sqrt{g_{00}}$ is the chronometrically invariant entropy density, while S^i is the chronometrically invariant entropy flux density. Denoting the first of these quantities by σ, we write

$$\sigma = \frac{\varphi}{1 - \frac{h_{ij}v^i v^j}{c^2}}, \quad S^i = \frac{\sigma v^i}{c}. \tag{11}$$

Using (8)-(11), we find a chronometrically invariant formulation of law (2)

$$\left\{\frac{{}^*\partial\sigma}{\partial t} + D\sigma + \left[\nabla_i(\sigma v^i) - \frac{F_i(\sigma v^i)}{c^2}\right]\right\}\delta\tau\delta V \geqslant \frac{\delta Q}{T}. \tag{12}$$

Using the chronometrically invariant operator for total time differentiation,

$$\frac{{}^*d}{dt} = \frac{{}^*\partial}{\partial t} + v^j\frac{{}^*\partial}{\partial x^j},$$

we can replace (12) by

$$\left\{\frac{{}^*d\sigma}{dt} + D\sigma + \sigma\left[{}^*\nabla_i v^i - \frac{F_i v^i}{c^2}\right]\right\}\delta\tau\delta V \geqslant \frac{\delta Q}{T}. \tag{13}$$

The second terms in Eqs. (12) and (13) (those containing D) take into account the dependence of the real change of the entropy density on the deformation of the reference frame; the quantities in brackets represent the physical divergence of the vectors σv^i and v^i. The term containing F_i, which represents the difference between the physical divergence and the mathematical divergence, takes into account the difference in the rate of time evolution at the boundaries of the volume element [2].

REFERENCES

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8 October 1969

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