FLUCTUATIONS IN THE PROPAGATION DIRECTION OF SCATTERED WAVES UPON REFLECTION FROM A LAYER CONTAINING RANDOM INHOMOGENEITIES

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Measurement of the approach angles of radial waves by differential-phase systems is based on the use of a phase gradient along the propagation direction or, more precisely, a difference between the phases of the fields at two (or several) spatial points. In this connection it is interesting to treat the problem of finding the distribution function and statistical characteristics of an arbitrary phase of the field upon reflection from an inhomogeneous plasma layer (such as the ionosphere).

After reflection from an inhomogeneous ionospheric layer, a wave propagates in a free half-space. The reflected field at any distance from the layer can be written as the sum of the average (regular) field $\bar{E}$ and the random (scattered) field ξ; if a plane, uniform harmonic wave $e^{-ik\sin\theta_0 x - ik\cos\theta_0 z}$ (x, z is the incidence plane, and the normal to the layer is along the z-axis) is incident on a layer, the scalar field E can be written as

$$E = \bar{E} + \xi = \bar{R}e^{-ik\sin\theta_0 x + ik\cos\theta_0 z} + \\ + \frac{1}{4\pi^2}\iint\limits_{-\infty}^{\infty} e^{-i\varkappa_1 x - i\varkappa_2 y + i\sqrt{k^2 - \varkappa_1^2 - \varkappa_2^2}\, z} c(\varkappa_1, \varkappa_2)\, d\varkappa_1\, d\varkappa_2, \tag{1}$$

where $\bar{R}$ is the complex reflection coefficient of the average field, and $c(\varkappa_1, \varkappa_2)$ is the Fourier component of scattered field ξ. Writing the total field E as

$$E = E_R + iE_i = Ae^{i\Psi}, \tag{2}$$

where $E_R = \mathrm{Re}E,\ E_i = \mathrm{Im}E$, A is the amplitude of the total field, and Ψ is the phase of the total field, and setting $\Psi = \varphi_0 - \varphi$, where φ_0 is the phase of the average field, we find

$$E_R = A\cos(\varphi_0 - \varphi) = A\cos\varphi\cos\varphi_0 + A\sin\varphi\sin\varphi_0,$$
$$E_i = A\sin(\varphi_0 - \varphi) = A\cos\varphi\sin\varphi_0 - A\sin\varphi\cos\varphi_0.$$

Using $A\cos\varphi = B,\ A\sin\varphi = C$, we find the following expression for the real and imaginary parts of the scattered field:

$$\xi_R = (B - \rho_0)\cos\varphi_0 + C\sin\varphi_0,$$
$$\xi_i = (B - \rho_0)\sin\varphi_0 - C\cos\varphi_0, \tag{3}$$

where $\rho_0 = |\bar{R}|$.

To determine the distribution function of an arbitrary phase $\varphi_x(\varphi_y, \varphi_z)$ we must know the combined four-dimensional distribution of the field amplitude and phase and their derivatives with respect to $x (y, z)$. The initial data used to solve this problem are the combined distribution $B_1 = B - \rho_0$, C, and their derivatives with respect to $x (y, z)$, B_{1x}, C_x [1]. Here we must know the combined distribution of the real and imaginary parts of the scattered field, ξ_R, ξ_i, and their derivatives with respect to $x (y, z)$, ξ_{Rx}, ξ_{ix}. It is difficult to solve this problem for the general case of an arbitrary distance from the layer, so we seek the distribution function φ_x in the Fraunhofer diffraction zone, assuming, on the basis of [2-4], that the scattered field in the Fraunhofer zone has a normal distribution. Then

B_1, C, B_{1x}, and C_x are also abnormal distributions, and we find from Eq. (3)

$$B_1 = \xi_R \cos\varphi_0 + \xi_i \sin\varphi_0, \quad C = \xi_R \sin\varphi_0 - \xi_i \cos\varphi_0;$$
$$B_{1x} = (\xi_{Rx} - k\sin\theta_0\xi_i)\cos\varphi_0 + (\xi_{ix} + k\sin\theta_0\xi_R)\sin\varphi_0, \tag{4}$$
$$C_x = (\xi_{Rx} - k\sin\theta_0\xi_i)\sin\varphi_0 - (\xi_{ix} + k\sin\theta_0\xi_R)\cos\varphi_0.$$

Assuming that there is no internal correlation of the field in the Fraunhofer diffraction zone, i.e., that

$$\overline{\xi_R\xi_i} = \overline{\xi_{Rx}\xi_{ix}} = \overline{\xi_R\xi_{Rx}} = \overline{\xi_i\xi_{ix}} = 0,$$

we find the combined four-dimensional normal distribution $W(B_1, C, B_{1x}, C_x)$. For this purpose we consider four quantities, defined by [5]

$$x_1 = B_1; \; x_2 = C; \quad x_3 = B_{1x} + \frac{\sigma_{B1x}}{\sigma_1}\widetilde{R}_{14}C; \quad x_4 = C_x - \frac{\sigma_{B1x}}{\sigma_1}\widetilde{R}_{14}B_1, \tag{5}$$

where

$$\overline{x_1^2} = \overline{x_2^2} = \overline{\xi_R^2} = \overline{\xi_i^2} = \sigma_1^2; \quad \overline{x_3^2} = \overline{x_4^2} = \sigma_{B1x}^2(1 - \widetilde{R}_{14}^2);$$
$$\sigma_{B1x}^2 = \overline{\xi_{Rx}^2} = \overline{\xi_{ix}^2} = \sigma_1^2 P_*^2; \quad \widetilde{R}_{14} = \frac{P^*}{P_*};$$
$$P^* = \frac{\iint_{-\infty}^{\infty} (\varkappa_1 - k\sin\theta_0) F(\varkappa_1, \varkappa_2)\, d\varkappa_1\, d\varkappa_2}{\iint_{-\infty}^{\infty} F(\varkappa_1, \varkappa_2)\, d\varkappa_1\, d\varkappa_2}; \quad P_*^2 = \frac{\iint_{-\infty}^{\infty} (\varkappa_1 - k\sin\theta_0)^2 F(\varkappa_1, \varkappa_2)\, d\varkappa_1\, d\varkappa_2}{\iint_{-\infty}^{\infty} F(\varkappa_1, \varkappa_2)\, d\varkappa_1\, d\varkappa_2}, \tag{6}$$

$F(\varkappa_1, \varkappa_2)$ is the average angular energy spectrum of scattered field ξ, and P_* and P^* characterize the mean square bandwidth for the energy spectrum and the asymmetry of the spectrum in the x, z incidence plane, respectively. We can show that the average values of the products satisfy

$$\overline{x_1x_2} = \overline{x_1x_3} = \overline{x_1x_4} = \overline{x_2x_3} = \overline{x_2x_4} = \overline{x_3x_4} = 0,$$

so the combined probability density is

$$W(x_1, x_2, x_3, x_4) = W(x_1)W(x_2)W(x_3)W(x_4),$$

where $W(x_1), ..., W(x_4)$ are the respective probability densities for $x_1, ..., x_4$. Since the functional determinant of transformation (5) is unity, we have the following for the unknown distribution function:

$$W(B_1, C, B_{1x}, C_x) = \frac{1}{(2\pi)^2\sigma_1^2\sigma_{B1x}^2(1-\widetilde{R}_{14}^2)} \exp\left\{-\frac{B_1^2 + C^2}{2\sigma_1^2} - \frac{1}{2\sigma_{B1x}^2(1-\widetilde{R}_{14}^2)}\left[\left(B_{1x} + \frac{\sigma_{B1x}}{\sigma_1}\widetilde{R}_{14}C\right)^2 + \left(C_x - \frac{\sigma_{B1x}}{\sigma_1}\widetilde{R}_{14}B_1\right)^2\right]\right\}. \tag{7}$$

Replacing variables in Eq. (7) according to

$$B_1 = B - \rho_0 = A\cos\varphi, \quad C = A\sin\varphi,$$
$$B_{1x} = A_x\cos\varphi - A\varphi_x\sin\varphi, \quad C_x = A_x\sin\varphi + A\varphi_x\cos\varphi, \tag{8}$$

we find the original four-dimensional distribution $W(A, \varphi, A_x, \varphi_x)$. Since the Jacobian of Transformation (8) is equal to A^2, we have

$$W(A, \varphi, A_x, \varphi_x) = \frac{A^2}{(2\pi)^2\sigma_1^2\sigma_{B1x}^2(1-\widetilde{R}_{14}^2)} \times \exp\left\{-\frac{1}{1-\widetilde{R}_{14}^2}\left[\frac{1}{2\sigma_1^2}(A^2 - 2A\rho_0\cos\varphi + \rho_0^2) + \frac{1}{2\sigma_{B1x}^2}(A_x^2 + A^2\varphi_x^2) + \frac{\widetilde{R}_{14}}{\sigma_1\sigma_{B1x}}(-A^2\varphi_x + A_x\rho_0\sin\varphi + A\rho_0\varphi_x\cos\varphi)\right]\right\}. \tag{9}$$

To find the distribution function of an arbitrary phase φ_x we must integrate Eq. (9) over A_x, A, φ. Omitting the lengthy calculations, we state the final result, which is an expression for the distribution function of the x derivative of the phase of the field in the Fraunhofer diffraction zone:

$$W(\varphi_x) = \frac{1}{(2)^{3/2}\sigma_1^3 P_* (1-\widetilde{R}_{14}^2)^{1/2}} \frac{e^{\frac{q^2}{4m}}}{4m^{3/2}} e^{-\frac{\beta^2}{1-\widetilde{R}_{14}^2}} \left[{}_1F_1\left(\frac{1}{2}, 1; -\chi\right) + \frac{q^2}{4m} {}_1F_1\left(\frac{1}{2}, 2; -\chi\right)\right], \tag{10}$$

where

$$\beta^2 = \frac{\rho_0^2}{2\sigma_1^2}, \quad q^2 = \frac{2\beta^2}{1-\widetilde{R}_{14}^2}\left(\frac{1}{\sigma_1} - \frac{\widetilde{R}_{14}}{\sigma_{B1x}}\varphi_x\right)^2,$$

$$m = \frac{1}{2(1-\widetilde{R}_{14}^2)}\left(\frac{1}{\sigma_1^2} - \frac{2\widetilde{R}_{14}\varphi_x}{\sigma_1\sigma_{B1x}} + \frac{\varphi_x^2}{\sigma_{B1x}^2}\right), \quad \chi = \frac{q^2}{4m} - \beta^2\frac{\widetilde{R}_{14}^2}{1-\widetilde{R}_{14}^2},$$

and ${}_1F_1$ is the degenerate hypergeometric function.

We see from Eq. (10) that with $P^*=0$, i.e., if the average angular energy spectrum is symmetric with respect to $k\sin\theta_0$, distribution $W(\varphi_x)$ is an even function, and the corresponding curve is symmetric with respect to $k\sin\theta_0$. If P* is nonvanishing, distribution $W(\varphi_x)$ is asymmetric. Let us evaluate the average value of the phase derivative $\bar{\varphi}_x$, for this case, i.e., the integral

$$\bar{\varphi}_x = \int_{-\infty}^{\infty} \varphi_x W(\varphi_x)\, d\varphi_x. \tag{11}$$

Substituting into Eq. (11) the expression for $W(\varphi_x)$, and changing the order of integration, we find

$$\bar{\varphi}_x = P^* e^{-\beta^2}. \tag{12}$$

Analogous expressions can be found for the distribution functions of the y and z derivatives of the field phase and the distribution functions of their average values $\bar{\varphi}_y, \bar{\varphi}_z$ (in this case, we have $\bar{\varphi}_y=0$).

We see from Eq. (12) that with $\beta^2=0$, we have $\bar{\varphi}_x=P^*$, and the $W(\varphi_x)$ distribution is central. In this case the average value of the x derivatives of the phase is equal to the average displacement of the angular energy spectrum of scattered field $F(\varkappa_1, \varkappa_2)$ with respect to $k\sin\theta_0$. As $\beta^2\to\infty$, we find $\bar{\varphi}_x\to 0$, and the average field direction coincides with that of the unperturbed field. Under real conditions, however, even with large β^2, the value of $\bar{\varphi}_x$ may be quite large, since practical measurements must be based on a finite sample of the random quantity. The effect of this latter circumstance is governed by the asymptotic behavior of $W(\varphi_x)$. According to Eq. (10), as $|\varphi_x|\to\infty$ we find $W(\varphi_x)\sim\frac{1}{|\varphi_x|^3}$. But then the transition from the infinite averaging interval (from the general sample) to a finite interval (to a finite sample) in the case of an asymmetric distribution can significantly change $\tilde{\varphi}_x$:

$$\tilde{\varphi}_x = \int_{-\Delta}^{\Delta} \varphi_x W(\varphi_x)\, d\varphi_x, \tag{13}$$

where Δ is the averaging interval. Using Eqs. (10) and (13) for the particular case $\beta^2\gg 1$, we find

$$\tilde{\varphi}_x = \frac{2}{\sqrt{\pi}} P^* \frac{\Delta^2}{P_*^3}\beta \exp\left[-\frac{\beta^2\frac{\Delta^2}{P_*^2}}{1+\frac{\Delta^2}{P_*^2}}\right]$$

Under the condition $\Delta \ll \frac{P_*^2}{2P^*}$. Furthermore, using Eq. (10), we can show that the probability of 0.95 and large β we have $\beta\,\Delta \approx 2\frac{P_*}{\beta}$ and thus

$$\tilde{\varphi}_x \approx \frac{0,32}{\sqrt{\pi}}\,\frac{P^*}{\beta^2}. \qquad (14)$$

In this case the average value of an arbitrary phase is on the same order of magnitude as β with an average shift of the angular energy spectrum of the total field [6] with respect to $k\sin\theta_0$, i.e., with

$$P_E^* = \frac{\iint\limits_{-\infty}^{\infty} (\varkappa_1 - k\sin\theta_0)\,F_E(\varkappa_1, \varkappa_2)\,d\varkappa_1\,d\varkappa_2}{\iint\limits_{-\infty}^{\infty} F_E(\varkappa_1, \varkappa_2)\,d\varkappa_1\,d\varkappa_2}, \qquad (15)$$

where F_E is the average angular energy spectrum of total field E. Since we have $F_E(\varkappa_1, \varkappa_2) = \rho_0^2\delta(\varkappa_1 - k\sin\theta_0)\,\delta(\varkappa_2) + F$, we find

$$P_E^* = \frac{P^*}{1+\beta^2} \qquad (16)$$

and with $\beta = 0$ we have $P_E^* = P^*$, or with $\beta^2 \gg 1$ we have $P_E^* \sim \frac{P^*}{\beta^2}$.

As an example, we find an expression for $\tilde{\varphi}_x$, using the results of [8], where P* was calculated for the case in which an inhomogeneous ionospheric layer having a dielectric constant

$$\varepsilon = \bar{\varepsilon}(z) + \mu(x, y, z),$$

where $\bar{\varepsilon}(z) = 1 - \frac{z}{z_1}$, and $\mu(x, y, z)$ is the homogeneous random Gaussian field, is struck at an angle θ_0 by a plane, uniform, harmonic wave $e^{-ik\sin\theta_0 x - ik\cos\theta_0 z}$. The problem was solved by the small-perturbation method, so the solution holds in the case $\beta^2 \gg 1$. In this case for large-scale inhomogeneities ($kl \gg 1$, where l is the correlation radius for the random inhomogeneities) and for sufficiently large incidence angles, we have

$$P^* \approx -\frac{1}{kl^2\sin\theta_0}$$

and, according to (14),

$$\tilde{\varphi}_x \approx -\frac{0,32}{\sqrt{\pi}}\,\frac{1}{kl^2\sin\theta_0\beta^2}. \qquad (17)$$

As was mentioned above, practical measurements of the incidence angles of differential-phase systems are based on the use of the difference between the phases of fields at two (or several) spatial points. It is usually assumed here that the difference between the field phases at two spatial points is related to the incidence angle of the radio waves by [7]

$$\Delta\Psi = kd\sin\theta_0,$$

where d is the distance between antennas and θ_0 is the incidence angle of the radio wave. Accordingly, the $\bar{\varphi}_x$ value calculated above represents the systematic error due to the wave propagation in a randomly inhomogeneous ionosphere. To evaluate the angular size of this error, i.e., to calculate the angle between the incidence direction of the (unperturbed) wave and the incidence direction corresponding to the average gradient in the phase of the total field, we write the average value of the derivative of the phase field with respect to x as

$$\bar{\Psi}_x = -k\sin\theta_0 - \bar{\varphi}_x = -k\sin(\theta_0 + \gamma), \qquad (18)$$

where γ is the unknown angle. If $\gamma \ll 1$, we find from (18)

$$\gamma \approx \frac{\bar{\varphi}_x}{k\cos\theta_0}.$$

Accordingly, during oblique probing of the ionosphere a systematic error may occur in the determination of the incidence angle of the radio waves, due to the asymmetry of the spectrum of reflected scattered waves in the incidence plane. For normal incidence of the wave on the layer, the $F(\varkappa_1, \varkappa_2)$ spectrum is symmetric with respect to $\varkappa_1 = 0, \varkappa_2 = 0$, and we have $\bar{\varphi}_x = 0$.

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