EVALUATION OF THE COLLISION INTEGRAL AND SOLUTION OF THE BASIC EQUATIONS OF ELECTROMAGNETIC CASCADE THEORY

A. A. Belyaev, I. P. Ivanenko, and A. A. Kirillov

Vestnik Moskovskogo Universiteta. Fizika,
Vol. 32, No. 2, pp. 29-33, 1977

UDC 549.121.7

A new analytic solution is presented for the one-dimensional problem of electromagnetic cascade theory, taking into account ionization losses and the Landau-Pomeranchuk effect. The ionization losses are not assumed to be proportional to the variation in the number of particles in the shower, but are taken into account by direct approximation of the collision integral, so that the specific properties of the medium are more accurately taken into account. The proposed approximation and the solution presented in this paper can be used to obtain the distribution functions with their prescribed accuracy. These solutions are valid in a broad energy range and yield improved versions of known distribution functions.

The difficulties that arise in the solution of the equations of cascade theory including ionization losses have forced many workers to use the well-known [1] approximation for the collision integral. The corresponding term in the cascade equation contains an empty derivative of the required function $P(E_0, E, t)$. This function gives the number of electrons with energy E at a depth t in the shower produced by a particle of energy E_0. Estimates of the precision of this approximation and the development of more accurate approximations for the collision integral as a differential of $P(E_0, E, t)$ encounter considerable difficulties [2]. In this paper, $W_{ion}(E, E')$ is the probability that an electron of energy E will lose energy E-E' per unit power length through the excitation and ionization of the atoms in the medium. As in [3], we use the following expression:

$$W_{ion}(E,E') = \begin{cases} c_1(E,z)\cdot\dfrac{1}{(E-E')^2} & E-E' > \varepsilon_l \\ c_2(E,z) & E-E' < \varepsilon_l , \end{cases} \quad (1)$$

where

$$c_1(E,z) = 0{,}153\,\frac{(E+mc^2)^2}{E(E+2mc^2)}\,\frac{z}{A}\,t \quad (\text{MeV, rad. u.})$$

z is the atomic number, A is the atomic weight, t is the radiation unit of length, $\varepsilon_l > I$, and I is the average ionization potential of the atom. It has been found [2-4] that this description of ionization is physically accurate for a transport process occurring in a region whose linear dimensions are much greater than the electron mean free path between collisions involving the excitation and ionization of the atoms in the medium. The critical energy β is regarded in [5] as the average value of (dE/dx)t, and this can be used to calculate $c_2(E, z)$ from the condition

$$\frac{dE}{dx}t = \int_0^E (E-E')\,W_{ion}(E,E')\,dE' = \beta .$$

Since dE/dx is a slowly varying function of the energy down to E ~ 0.5 MeV, the use of $c_2(z)$ in (1) leads to an error of less than a few percent in (dE/dx)t or $E \gtrsim 0.5$ MeV.

The Landau-Pomeranchuk effect is taken into account in the present work in the same way as in [6]: we use the following approximations for $W_p(E, E')dE$ and $W_e(E, E')dE$:

$$W_e(E', E)\,dE = \begin{cases} \frac{dE}{E} & E' < E_k^e \\ (\alpha_1 - \alpha_2 \ln E')\frac{dE}{E} & E' \geqslant E_k^e \end{cases},$$

$$W_p(E, E')\,dE = \begin{cases} \sigma_0 dE & E' < E_k^p \\ (\beta_1 - \beta_2 \ln E') & E' \geqslant E_k^p \end{cases},$$

$$\alpha_1 = 3{,}85, \quad \alpha_2 = 0{,}106, \quad E_k^e = 4{,}75 \cdot 10^6\ \text{MeV}, \quad \beta_1 = 4{,}025,$$

$$\beta_2 = 0{,}1085, \quad E_k^p = 1{,}04 \cdot 10^7\ \text{MeV}, \quad \sigma_0 = 0{,}773.$$

The influence of the effect for certain $E \gg \beta$ and $E \ll E_k^e$ was investigated in [7] but without including ionization losses.

The problem has been solved by the "q-method" proposed in [8] and developed further in [9]. In the present approximation, the system of equations describing the development of the shower initiated by a primary electron can be written in the form [1]:

$$\frac{\partial P(E_0, E, t)}{\partial t} = -P(E_0, E, t)\left(\int_0^E W_e(E, E')\,dE' + \int_0^{\frac{E}{2}} W_{\text{ion}}(E, E')\,dE'\right) +$$

$$+\int_E^{E_0} P(E_0, E, t)(W_e(E', E' - E) + W_{\text{ion}}(E', E))\,dE' +$$

$$+2\int_E^{E_0} \Gamma(E_0, E', t)\,W_p(E, E')\,dE',$$

$$\frac{\partial \Gamma(E_0, E, t)}{\partial t} = \int_E^{E_0} P(E_0, E', t)\,W_e(E, E')\,dE' - \Gamma(E_0, E, t)\int_0^E W_p(E, E')\,dE'.$$

Taking the Laplace transform with respect to the depth t and eliminating the transform $\Gamma(E_0, E, \lambda)$ of the photon distribution function, we obtain

$$L_{\text{ion}}[P(E_0, E, \lambda)] = \delta(E_0 - E). \tag{2}$$

Next, as usual, we introduce the parameter s and reduce the integral $\int_E^{E_0} E'^s L_{\text{ion}}[P(E_0, E', \lambda)]\,dE'$ to the form

$$\int_E^{E_0} E'^s L_{\text{ion}}[P(E_0, E', \lambda)]\,dE' = \int_E^{E_0} P(E_0, E', \lambda)\,\varphi_{\text{ion}}(E', E, \lambda, s)\,dE', \tag{3}$$

where

$$\varphi_{\text{ion}}(E', E, \lambda, s) = \varphi(E', E, \lambda, s) + \Delta\varphi_{\text{ion}}(E', E, s),$$

$$\Delta\varphi_{\text{ion}} = -\int_E^{E} E''^s W_{\text{ion}}(E' - E'')\,dE'' + E'^s \int_0^{E'} W_{\text{ion}}(\varepsilon)\,d\varepsilon, \quad \sigma(\varepsilon) = \int_0^{\varepsilon} W_p(\varepsilon, E)\,dE,$$

and the detailed form of $\varphi(E, E', \lambda, s)$ is given by (8) and (9). We now define $q(E_0, E, \lambda, s)$ by

$$\int_E^{E_0} P(E_0, E', \lambda)\frac{\varphi_{\text{ion}}(E, E', \lambda, s)}{E^s}\,dE' = q(E_0, E, \lambda, s)\int_E^{E_0} P(E_0, E', \lambda)\,dE',$$

and, using (2) and (3), we obtain the following Volterra equation of the first kind for $q(E_0, E, \lambda, s)$:

$$\int_E^{E_0} \left[s\left(\frac{E_0}{E'}\right)^s \frac{1}{E'}\frac{1}{q} + \left(\frac{E_0}{E'}\right)^s \frac{\frac{\partial}{\partial E'} q}{q^2} \right] \frac{\varphi_{\text{ion}}(E, E', \lambda, s)}{E^s}\, dE' - \left(\frac{E_0}{E}\right)^s = 0. \tag{4}$$

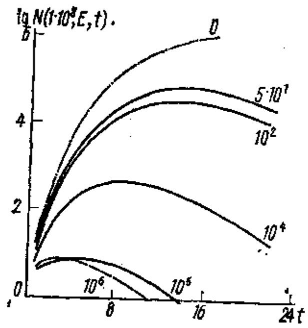

Fig. 1. Broken curve shows the Bethe-Heitler cross sections without the Landau-Pomeranchuk effect [8].

For $\Delta\varphi_{ion}=0$ (without ionization losses), a sufficiently accurate solution of (4) with and without the Landau-Pomeranchuk effect was obtained in [7] and [9], respectively. In these solutions $E < \beta\cdot 10^2$ and $q(E_0, s)$ is independent of E. The ionization losses can be neglected for $E > \beta\cdot 10^2$, so that $q(E, s)$ must be determined for $E \leq \beta\cdot 10^2$ if ionization losses are to be taken into account. This can be used to construct the distribution function with the above approximation for the collision integral both with and without the Landau-Pomeranchuk effect. As noted in [9], the approach to the solution of (4) used there provides a simple means of including additional processes because the inclusion of such processes leads to linear transformations of the kernel of the equation and the corresponding transformations of the discrepancy that results after the substitution of the approximate solution into the equation. In fact, if we use the solutions given in [7-9] as the zero-order approximation, we obtain the following expressions for the discrepancy $\Delta_{ion}(q(s))$:

$$\Delta_{\text{ion}}(q(s)) = \int_E^{E_0} \left[s\left(\frac{E_0}{E'}\right)^s \frac{1}{E'}\frac{1}{q(s)} \right] \left[\frac{\varphi(E', E, \lambda, s)}{E^s} + \frac{\Delta\varphi_{\text{ion}}(E', E, \lambda, s)}{E^s} \right] dE' - \left(\frac{E_0}{E}\right)^s = \int_E^{E_0} s\left(\frac{E_0}{E'}\right)^s \frac{1}{E'}\frac{1}{q(s)} \frac{\Delta\varphi_{\text{ion}}}{E^s}\, dE' + \Delta(q(s)). \tag{5}$$

If we evaluate the next approximation from the formula

$$q_{i+1} = q_i \left(1 + \left(\frac{E}{E_0}\right)^s \cdot \Delta_{\text{ion}}(q_i)\right), \tag{6}$$

we obtain the first approximation $q_1(E_0, E, s)$. From (5) and (6) we obtain the following expression for $\Delta q(E, s) = q_1(E_0, E, s) - q(s)$ when $E < \beta\cdot 10^2$:

$$\Delta q(E, s) = \left(\frac{1}{E_0}\right)^s \int_E^{E_0} s\left(\frac{E_0}{E'}\right)^s \cdot \frac{1}{E'}\, \Delta\varphi_{\text{ion}}(E', E, \lambda, s)\, dE'. \tag{7}$$

Calculations based on (7) show that $\Delta q(s,\beta) \leqslant s q(s)$ and $\Delta q(s, E)$ is negligible for $E \gg \beta\cdot 10^2$, so that the error in the first approximation does not exceed a few percent for $E \geq \beta$.

To improve the solution of (4) for $E < \beta$, we can specify a functional form for $q(E_0, E, s)$. Suppose that

$$q(E, s) = q_0(s) + \frac{1}{a(s) + b(s)E^{\frac{s}{2}} + c(s)E^s}. \tag{8}$$

Table 1

s	0,2	0,4	0,6	0,8	1,0	1,2	1,4	1,6	1,8
$b(s)$	−1,6	−0,2	−0,03	0,021	0,025	0,023	0,0195	0,015	0,009
$c(s)$	2,65	0,7	0,3	0,12	0,062	0,03	0,015	0,007	0,005

Substituting (8) in (5), we then obtain the following expression for the discrepancy:

$$\begin{aligned}\Delta(q(E,s)) \approx & \int_E^1 s\left(\frac{E_0}{E'}\right)^s \frac{1}{E'} a(s)\left(1+\frac{b(s)}{2a(s)} E'^{\frac{s}{2}}\right)\frac{\varphi_{ion}}{E^s}\, dE' + \\ & + \int_1^\beta s\left(\frac{E_0}{E'}\right)^s \frac{1}{E'} q(s)\left(c^2(s) E'^{2s} + \frac{3}{2} b(s) c(s) E'^{\frac{3}{2}s} + \right. \\ & \left. + \left(\frac{b^2(s)}{2} + a(s)c(s)\right)\right) E'^s \frac{\varphi_{ion}}{E^s}\, dE' + \\ & + \int_\beta^{E_0} s\left(\frac{E_0}{E'}\right)^s \frac{1}{E'}\frac{1}{q(s)}\frac{\varphi_{ion}}{E^s}\, dE' - \left(\frac{E_0}{E}\right)^s\end{aligned} \qquad (9)$$

and by suitably choosing $a(s)$, $b(s)$, $c(s)$ we can minimize this discrepancy in the required range. To simplify the calculations, we note that for sufficiently low energies E, we have $1 + aq_0 \gg q_0(b(s)E^{s/2} + c(s)E^s)$. The expression for the discrepancy is then practically independent of $c(s)$. Moreover, it is clear from (8) that the dependence on $c(s)$ is strongest for $E \approx \beta$. At the same time, $\Delta q(E, s)/E^s$ is practically constant for $\beta \ll E \leq \beta \cdot 10^2$, so that we can assume, approximately, that $c(s) = \Delta q(E, s)/\beta^s$. The parameter $a(s)$ has an effect for $E \ll \beta$, but calculations show that it can be neglected in the physically interesting region. In view of the foregoing, we can use (9) both for the direct evaluation of $b(s)$ and for estimating the precision of the selected approximation to the solution.

Figure 1 shows the integral distribution functions for shower electrons due to a primary electron of energy $E_0 = 1 \cdot 10^8$ MeV in a dense absorber (z = 82). The threshold energies are shown against the curves (in MeV). It was assumed that $c_1(82) = 11 \cdot 10^4$, $c_2(82) = 0.384$ and the values of $b(s)$ and $c(s)$ were taken to be as shown in Table 1.

It is clear from the figure that the inclusion of the ionization losses is important at low cutoff thresholds and its influence increases with increasing depth in the shower. In this region, the ionization losses act in opposition to the Landau-Pomeranchuk effect although they do not compensate this effect because the two act on different shower components: ionization losses cut off the soft component of the shower whereas the Landau-Pomeranchuk effect describes the increase in the penetrating power of the hard component.

REFERENCES

1. Rossi and Bruno, High-Energy Particles, New York, 1952.
2. V. A. Kuzyuk and A. Kh. Rakhmatulina, Zhurn. vychislit. matem. i matem. fiziki, vol. 15, p. 1248, 1975.
3. A. A. Belyaev, V. V. Guzhavin, and I. P. Ivanenko, Preprint FIAN, no. 34, 1975.
4. H. Bichsel and R. P. Saxon, Phys. Rev., vol. All, p. 1287, 1975.
5. S. Z. Belen'kii, Shower Processes in Cosmic Rays [in Russian], Moscow, 1948.
6. V. V. Guzhavin, I. P. Ivanenko, and T. M. Roganova, Preprint FIAN, no. 61, 1975.
7. V. V. Guzhavin, I. P. Ivanenko, A. A. Kirillov, and T. M. Roganova, Proc. of 14th Int. Cosmic Ray Conf., vol. 7, p. 2609, 1975.
8. I. P. Ivanenko, Electromagnetic Cascade Processes [in Russian], Moscow University Press, 1972.
9. I. P. Ivanenko, A. A. Kirillov, and Yu. I. Paskhalov, Vestn. Mosk. un-ta, Fiz. astron. [Moscow University Physics Bulletin], no. 5, p. 598, 1976.

28 June 1976

Scientific-Research Institute for Nuclear Physics