DIFFERENTIAL EQUATIONS FOR THE NONANGULAR ELEMENTS OF A HYPERBOLIC ORBIT

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Differential equations are presented for the nonangular elements (semimajor axis, eccentricity, and inclination) of an intermediate hyperbolic orbit of a mass point moving in the gravitational field of a condensed planet, subject to a perturbation by another planet. The intermediate orbit is based on the symmetric variant of the problem of two fixed centers. The perturbation function is taken in the form of the Hill term in the expansion of the potential due to the perturbing planet, corrected for the elliptical character of its motion. The right-hand sides of the equations are in the form of trigonometric series whose arguments are combinations of the angular elements of the intermediate motion of the mass point and the angular elements of the perturbing body, whereas the amplitudes depend on the nonangular elements of the two and are series in powers of the small parameter of the problem of two fixed centers.

In our previous paper [1] we derived the differential equations for the osculating elements a, e, s, ω, Ω, M and the orbit parameter p [2, 3] of the hyperbolic trajectory in the symmetric variant of the problem of two fixed centers [4]. The elements a, e, s and p were referred to as the nonangular elements in [1]. The differential equations for these elements can be written in the following general form [1]:

$$\frac{da}{dt} = \dot{\hat{a}}[1 + 3\varepsilon^2(\widehat{e^2} - 1)(1 - \widehat{s^2})] + \dot{\hat{e}}\cdot 10\varepsilon^2\hat{a}\hat{e}(1 - \widehat{s^2}) - 2\dot{\hat{s}}\varepsilon^2\hat{a}\hat{s}(\widehat{e^2} - 1),$$

$$e\frac{de}{dt} = -\frac{\dot{\hat{a}}}{\hat{a}}\varepsilon^2(\widehat{e^2} - 1)(3\widehat{e^2} + 1)(1 - \widehat{s^2}) + \dot{\hat{e}}\hat{e}[1 -$$

$$-\varepsilon^2 12\widehat{e^2}(1 - \widehat{s^2})] + \dot{\hat{s}}\varepsilon^2\hat{s}(\widehat{e^2} - 1)(3\widehat{e^2} + 1), \tag{1}$$

$$s\frac{ds}{dt} = -\frac{\dot{a}}{\hat{a}}\varepsilon^2\widehat{s^2}(1 - \widehat{s^2})(\widehat{e^2} - 1) - \dot{\hat{e}}3\varepsilon^2\widehat{s^2}(1 - \widehat{s^2})\hat{e} + \dot{\hat{s}}\hat{s}[1 - \varepsilon^2(\widehat{e^2} - 1)],$$

$$\frac{dp}{dt} = \dot{\hat{p}}[1 - 6\varepsilon^2\widehat{\alpha^2}(\widehat{e^2} + 1)] - 4\dot{\hat{s}}\varepsilon^2\hat{a}\hat{s}(\widehat{e^4} - 1),$$

where the right-hand sides contain expressions for the derivatives of the quantities $\hat{a}$, $\hat{e}$, $\hat{s}$. The differential equations of the latter, given in terms of the first derivatives, are [1]:

$$\dot{\hat{a}} = 2\widehat{a^2}(\dot{x}F_x' + \dot{y}F_y' + zF_z'),$$

$$\dot{\hat{e}} = F_x'\cdot[x] + F_y'\cdot[y] + F_z'\cdot[z], \tag{2}$$

$$\dot{\hat{s}} = \frac{1 - \widehat{s^2}}{\hat{s}\hat{p}}\{F_x'[x'] + F_y'[y'] + F_z'[z'],$$

where

$$F_q' = \frac{1}{fm}\cdot\frac{\partial R}{\partial q}, \ (q = x, y, z),$$

R is the perturbation function,

$$[x]=\frac{\widehat{p}}{\widehat{e}}\dot{x}-\frac{1}{\widehat{a}\widehat{e}}r(\dot{x}r-x\dot{r}),$$

$$[y]=\frac{\widehat{p}}{\widehat{e}}\dot{y}-\frac{1}{\widehat{a}\widehat{e}}r(\dot{y}r-y\dot{r}),$$

$$[z]=\frac{\widehat{a}\widehat{p}-c^2}{\widehat{a}\widehat{e}}\dot{z}-\frac{1}{\widehat{a}\widehat{e}}r(\dot{z}r-z\dot{r}),$$

$$[x']=r(\dot{x}r-x\dot{r})-\frac{y}{1-\widehat{s}^2}(\dot{x}y-x\dot{y}),$$

$$[y']=r(\dot{y}r-y\dot{r})+\frac{x}{1-\widehat{s}^2}(\dot{x}y-x\dot{y}),$$

$$[z']=r(\dot{z}r-z\dot{r})-c^2\dot{z};$$

f is the gravitational constant; m is the mass of the central body; $r=\sqrt{x^2+y^2+z^2}$ is the radius vector; x, y, z are the q-coordinates of the point moving on the hyperbolic orbit; t is the time; $\dot{q}$ = dq/dt and $\dot{r}$ = dr/dt; c is the constant in the problem of two fixed centers [4]; and $\alpha^2 = 1 - s^2$.

We shall take the perturbation function in the form

$$R=\frac{fm'}{r'}\left[1+\frac{1}{2}\left(\frac{r}{r'}\right)^2(3\cos^2 H-1)\right],\ \cos H=\frac{xx'+yy'+zz'}{rr'}, \tag{3}$$

i.e., we shall suppose that the perturbation due to a body of mass m' and coordinates q': x', y', z', $r' = a'(1 - e'^2)/1 + e' \cos v'$ is the radius vector of the perturbing body (so that the body moves on the Kepler ellipse elements a', e', i', ω', Ω', M'), α' = = cos i', s' = sin i', and v' = u' - ω' is the true anomaly of the perturbing body. We recall that the perturbed mass point moves on a hyperbola with non-Keplerian osculating elements a, e, s, ω, Ω, M.

Consider a Cartesian set of coordinates connects to the central axially symmetric condensed body so that the origin of coordinates lies at the center of mass, the Oz axis lies along the axis of rotation, the Ox axis passes through the point of intersection of the equator of the body and the ecliptic, and the xy plane is equatorial. The coordinates of the mass point moving in the gravitational field of the central body and the body of mass m' that produces the perturbation can be expressed in terms of the corresponding orbit elements as follows:

$$x=r\widetilde{\alpha},\qquad x'=r'\widetilde{\alpha}',$$

$$y=r\widetilde{\beta},\qquad y'=r'\widetilde{\beta}',$$

$$z=r\widetilde{\gamma},\qquad z'=r'\widetilde{\gamma}';$$

where

$$\widetilde{\alpha}=-\sin\varphi\sin\Omega+\alpha\cos\varphi\cos\Omega,$$

$$\widetilde{\beta}=\sin\varphi\cos\Omega+\alpha\cos\varphi\sin\Omega,$$

$$\widetilde{\gamma}=\frac{\xi}{r}s\cos\varphi,$$

$$\xi=\frac{a(1-e^2)}{1+\bar{e}\cos\psi}[1-\varepsilon^2e^2(1-2s^2)+\cos\psi\varepsilon^2e^2(1-2s^2)],$$

$$\bar{e}=e[1-\varepsilon^2(e^2-1)(1-2s^2)];$$

ψ is the analog of the true anomaly; ε is the small parameter of the problem of two fixed centers, $\varphi=u+\bar{\omega}(\psi)$, $u=\psi+\omega$, [1, 3],

$$\widetilde{\alpha}' = \cos u' \cos \Omega' - \alpha' \sin u' \sin \Omega',$$
$$\widetilde{\beta}' = \cos u' \sin \Omega' + \alpha' \sin u' \cos \Omega',$$
$$\widetilde{\gamma}' = s' \sin u'.$$

We now use the notation

$$(r\dot{r}) = x\dot{x} + y\dot{y} + z\dot{z},$$
$$(rr') = xx' + yy' + zz',$$
$$(\dot{r}r') = \dot{x}x' + \dot{y}y' + \dot{z}z',$$
$$[r\dot{r}r'] = (x\dot{y} - \dot{x}y)(xy' - x'y) + (y\dot{z} - \dot{y}z)(yz' - y'z) + (z\dot{x} - \dot{z}x)(zx' - z'x).$$

In this notation, the perturbing function R (3) and its derivatives with respect to the coordinates of the perturbed point, F_q (2), can be transformed so that Eqs. (2) for $\hat{a}$, $\hat{e}$, $\hat{s}$, and $\hat{p}$ assume the form

$$\frac{d\hat{a}}{dt} = \frac{2\hat{a}^3}{r'^5}\{3(rr')(\dot{r}r') - r'^2(r\dot{r})\},$$
$$\frac{d\hat{e}}{dt} = \frac{1}{\hat{a}\hat{e}r'^5}\{r'^2[c^2z\dot{z} - \hat{a}\hat{p}(r\dot{r})] - 3(rr')[c^2z'\dot{z} - \hat{a}\hat{p}(\dot{r}r')] + 3(rr')[r\dot{r}r']\}, \quad (4)$$
$$\frac{d\hat{s}}{dt} = \frac{1-\hat{s}^2}{\hat{p}\hat{s}r'^5}\Big\{c^2z\dot{z}r'^2 - 3(rr')c^2\dot{z}z' - \frac{3}{\hat{a}^2}(rr')(x\dot{y} - \dot{x}y) \times$$
$$\times (xy' - x'y) + 3(rr')[r\dot{r}r']\Big\}$$

and

$$\frac{d\hat{p}}{dt} = \frac{1}{r'^5}\{6c^2\dot{z}z'(rr') - 2c^2z\dot{z}r'^2 - 6(rr')[r\dot{r}r']\}.$$

As before [1], we confine our attention to quantities of order ε^2. It then turns out that, after some relatively laborious transformations, the right-hand sides of (4) can be written in the form of trigonometric series with arguments ψ, v', ω, ω', Ω, Ω', in which the first two are directly connected with the time, or are the true anomalies of the perturbed point and the perturbing body, and the coefficients of these trigonometric functions, which depend on the nonangular elements a, a', e, e', s, s' and ε, i.e., equations (4), assume the form

$$\frac{d\hat{a}}{dt} = L_{\hat{a}} \sum_{\substack{k,l,m\\ i,j}} \{{}^{\hat{a}}A_{2i,j}^{2k,2l,m} \sin[2k(\Omega' - \Omega) \pm 2l\omega' \pm 2i\omega \pm j\psi \pm mv'] +$$
$$+ {}^{\hat{a}}B_{2i,j}^{1,2l,m} \cos(\Omega' - \Omega \pm 2l\omega' \pm 2i\omega \pm j\psi \pm mv')\},$$
$$\frac{d\hat{e}}{dt} = L_{\hat{e}} \sum_{\substack{k,l,m\\ i,j}} \{{}^{\hat{e}}A_{2i,j}^{2k,2l,m} \sin[2k(\Omega' - \Omega) \pm 2l\omega' \pm 2i\omega \pm j\psi \pm mv'] +$$
$$+ {}^{\hat{e}}B_{2i,j}^{1,2l,m} \cos(\Omega' - \Omega \pm 2l\omega' \pm 2i\omega \pm j\psi \pm mv')\},$$
$$\frac{d\hat{s}}{dt} = L_{\hat{s}} \sum_{\substack{k,l,m\\ i,j}} \{{}^{\hat{s}}A_{2i,j}^{2k,2l,m} \sin[2k(\Omega' - \Omega) \pm 2l\omega' \pm 2i\omega \pm j\psi \pm mv'] +$$
$$+ {}^{\hat{s}}B_{2i,j}^{1,2l,m} \cos(\Omega' - \Omega \pm 2l\omega' \pm 2i\omega \pm j\psi \pm mv')\}$$

and

$$\frac{d\hat{p}}{dt} = L_{\hat{p}} \sum_{\substack{k,l,m\\ i,j}} \{{}^{\hat{p}}A_{2i,j}^{2k,2l,m} \sin[2k(\Omega' - \Omega) \pm 2l\omega' \pm 2i\omega \pm j\psi \pm mv'] +$$

$$+ \widehat{{}^{p}B}{}^{1,2l,m}_{2k,j} \cos(\Omega' - \Omega \pm 2l\omega' \pm 2i\omega \pm j\psi \pm mv')\}. \tag{5}$$

The subscripts k, l, m, i, j take on the following values:

$$k = 0, 1;\ l = 0, 1;\ m = 0, 1, 2, 3, 4, 5;$$
$$i = 0, 1, 2;\ j = 0, 1, 2, 3, 4, 5, 6, 7;$$

the coefficients of the sums, i.e., $L_{\hat{a}}$, $L_{\hat{e}}$, $L_{\hat{s}}$, $L_{\hat{p}}$, are given by

$$L_{\hat{a}} = \frac{a^2 \sqrt{e^2 - 1}\, n}{16 a'^2 (1 - e'^2)^3} \frac{2\hat{a}^2}{1 + \bar{e}\cos\psi},$$
$$L_{\hat{e}} = \frac{a^2 \sqrt{e^2 - 1}\, n}{64 a'^2 (1 - e'^2)^3} \frac{\hat{a}(\hat{e}^2 - 1)}{\hat{e}(1 + \bar{e}\cos\psi)^2}, \tag{6}$$
$$L_{\hat{s}} = \frac{a^2 \sqrt{e^2 - 1}\, n}{64 a'^2 (1 - e'^2)^3} \frac{a^2 (e^2 - 1)^2}{\hat{p}\hat{s}(1 + \bar{e}\cos\psi)^2}$$

and

$$L_{\hat{p}} = -\frac{a^2 \sqrt{e^2 - 1}\, n}{32 a'^2 (1 - e'^2)^3} \frac{a^2 (e^2 - 1)^2}{(1 + \bar{e}\cos\psi)^2}.$$

The expressions for the coefficients of the trigonometric functions inside the sums are two unwieldy to be written out here (we have obtained these formulas to within terms of the order of ε^2). To get some idea as to the form of these coefficients, we reproduce below the coefficients in front of the constant terms (which do not directly depend on the time) for all the equations in (5):

$$\widehat{{}^{\varepsilon}A}{}^{000}_{20} = -2(2 - 3s'^2)(3e'^2 + 2)s^2e^2 \Big\{3 -$$
$$- \frac{\varepsilon^2}{8}[74e^2 - 104 - 3s^2(43e^2 - 53)]\Big\},$$
$$\widehat{{}^{\varepsilon}A}{}^{020}_{\pm 20} = \mp \frac{9}{2} s'^2 e'^2 s^2 e^2 \Big\{3 - \frac{\varepsilon^2}{8}[74e^2 - 104 - 3s^2(43e^2 - 53)]\Big\};$$
$$\widehat{{}^{\varepsilon}B}{}^{100}_{00} = \frac{3}{2}\varepsilon^2 \alpha' s'(3e'^2 + 2)s[8(e^4 + e^2 - 2) - s^2(9e^4 - 12e^2 - 32)],$$
$$\widehat{{}^{\varepsilon}B}{}^{100}_{\pm 20} = -12\alpha' s'(3e'^2 + 2)(1 \mp \alpha)se^2 \times$$
$$\times \Big\{1 + \frac{\varepsilon^2}{8}[18e^2 - 11 \pm \alpha(e^2 + 6) - \alpha^2(43e^2 - 53)]\Big\},$$
$$\widehat{{}^{\varepsilon}B}{}^{1\pm 20}_{00} = \mp \frac{9}{8}\varepsilon^2(1 \pm \alpha')s'e'^2 s[8(e^4 + e^2 - 2) - s^2(9e^4 - 12e^2 - 32)],$$
$$\widehat{{}^{\varepsilon}B}{}^{1\underset{[-]}{+}20}_{\pm 20} = \underset{[-]}{+} 9(1 \underset{[-]}{+} \alpha')s'e'^2(1 \mp \alpha)se^2 \times$$
$$\times \Big\{1 + \frac{\varepsilon^2}{8}[18e^2 - 11 \pm \alpha(e^2 + 6) - \alpha^2(43e^2 - 53)]\Big\}.$$

Here and henceforth the different signs refer to different signs in the arguments. For example, the coefficients $A^{2\underset{[-]}{+}20}_{\pm 20}$ and $B^{1\underset{[-]}{+}20}_{\pm 20}$ are those in front of the functions $\sin 2(\Omega' - \Omega \underset{[-]}{+} \omega' \pm \omega)$ and $\cos(\Omega' - \Omega \underset{[-]}{+} 2\omega' \pm 2\omega)$.

$$\widehat{{}^{\varepsilon}A}{}^{200}_{00} = -\frac{3}{4}\varepsilon^2 s'^2(3e'^2 + 2)s^2\alpha(9e^4 - 12e^2 - 32),$$
$$\widehat{{}^{\varepsilon}A}{}^{200}_{\pm 20} = \pm 3s'^2(3e'^2 + 2)(1 \mp \alpha)^2 e^2 \times$$

$$\times \left\{1 + \frac{\varepsilon^2}{8}[17e^2 + 11 \pm 2\alpha(e^2+6) - \alpha^2(43e^2 - 53)]\right\},$$

$${}^{e}\hat{A}_{00}^{2\pm 20} = -\frac{9\varepsilon^2}{16}(1 \pm \alpha')^2 e'^2 s^2 \alpha (9e^4 - 12e^2 - 32),$$

$${}^{e}\hat{A}_{\pm 20}^{2\overset{+}{[-]}20} = \pm \frac{9}{4}(1 \underset{[-]}{+} \alpha')^2 e'^2 (1 \mp \alpha)^2 e^2 \times$$

$$\times \left\{1 + \frac{\varepsilon^2}{8}[17e^2 + 11 \pm 2\alpha(e^2+6) - \alpha^2(43e^2 + 53)]\right\};$$

$${}^{s}\hat{A}_{20}^{000} = \varepsilon^2 (2 - 3s'^2)(3e'^2 + 2) s^2 \alpha^2 (2 - 3s^2) e^2,$$

$${}^{s}\hat{A}_{\pm 20}^{020} = \pm \frac{9}{4}\varepsilon^2 s'^2 e'^2 s^2 (2 - 3s^2)\alpha^2 e^2,$$

$${}^{s}\hat{A}_{00}^{200} = 3s'^2(3e'^2 + 2)s^2\alpha\left\{4 + \frac{\varepsilon^2}{2}[25e^2 + 12 - \alpha^2(45e^2 - 20)]\right\},$$

$${}^{s}\hat{A}_{\pm 20}^{200} = -\frac{3}{4}\varepsilon^2 s'^2(3e'^2+2)\alpha(1 \mp \alpha)^2(1 \pm \alpha)(5 \mp \alpha + 2\alpha^2)e^2,$$

$${}^{s}\hat{A}_{00}^{2\pm 20} = \frac{9}{4}(1 \pm \alpha')^2 e'^2 \alpha s^2 \left\{4 + \frac{\varepsilon^2}{2}[25e^2 + 12 - \alpha^2(45e^2 - 20)]\right\},$$

$${}^{s}\hat{A}_{\pm 20}^{2\overset{+}{[-]}20} = -\frac{9\varepsilon^2}{16}(1 \underset{[-]}{+} \alpha')^2 e'^2 \alpha (1 \mp \alpha)^2 (1 \pm \alpha)(5 \mp \alpha + 2\alpha^2) e^2;$$

$${}^{s}\hat{B}_{00}^{100} = -2\alpha'(3e'^2 + 2)\alpha^2\left\{1 + \frac{\varepsilon^2}{8}[23e^2 + 8 - 5\alpha^2(9e^2 - 4)]\right\},$$

$${}^{s}\hat{B}_{\pm 20}^{100} = \pm \frac{\varepsilon^2}{8}\alpha'(3e'^2 + 2)\alpha(1 \mp \alpha)(3 \pm \alpha + \alpha^2 \pm 4\alpha^3)e^2, \qquad (7)$$

$${}^{s}\hat{B}_{00}^{1\pm 20} = \pm \frac{3}{2}(1 \pm \alpha')e'^2\alpha^2\left\{1 + \frac{\varepsilon^2}{8}[23e^2 + 8 - 5\alpha^2(9e^2 - 4)]\right\},$$

$${}^{s}\hat{B}_{\pm 20}^{1\overset{+}{[-]}20} = \underset{[\pm]}{\mp} \frac{3\varepsilon^2}{32}(1 \underset{[-]}{+} \alpha')e'^2\alpha(1 \mp \alpha)(3 \pm \alpha + \alpha^2 \pm 4\alpha^3)e^2;$$

$${}^{p}\hat{A}_{20}^{000} = \varepsilon^2(2 - 3s'^2)(3e'^2 + 2)s^2(2 - 3s^2)e^2,$$

$${}^{p}\hat{A}_{\pm 20}^{020} = \pm \frac{9}{4}\varepsilon^2 s'^2 e'^2 s^2 (2 - 3s^2) e^2,$$

$${}^{p}\hat{A}_{00}^{200} = -9\varepsilon^2 s'^2 (3e'^2 + 2) s^2 \alpha (e^2 + 2),$$

$${}^{p}\hat{A}_{\pm 20}^{200} = \mp \frac{3}{2}\varepsilon^2 s'^2 (3e'^2 + 2)(1 \mp \alpha)^2 (1 + \alpha^2) e^2,$$

$${}^{p}\hat{A}_{00}^{2\pm 20} = -\frac{27}{4}\varepsilon^2 (1 \pm \alpha')^2 e'^2 s^2 \alpha (e^2 + 2),$$

$${}^{p}\hat{A}_{\pm 20}^{2\overset{+}{[-]}20} = \mp \frac{9}{8}\varepsilon^2 (1 \underset{[-]}{+} \alpha')^2 e'^2 (1 \mp \alpha)^2 (1 + \alpha^2) e^2;$$

$${}^{p}\hat{B}_{00}^{100} = \frac{\varepsilon^2}{2}\alpha'(3e'^2 + 2)(2 - 3s^2)(e^2 + 2),$$

$${}^{p}\hat{B}_{\pm 20}^{100} = \frac{\varepsilon^2}{2}\alpha'(3e'^2 + 2)\alpha^2(1 \mp \alpha)e^2,$$

$${}^{p}\hat{B}_{00}^{1\pm 20} = \mp \frac{3}{8}\varepsilon^2(1 \pm \alpha')e'^2(2 - 3s^2)(e^2 + 2),$$

$${}^{p}\hat{B}_{\pm 20}^{1\overset{+}{[-]}20} = \mp \frac{3}{8}\varepsilon^2(1 \underset{[-]}{+} \alpha')e'^2(1 \mp \alpha)\alpha^2 e^2.$$

The explicit form of (5) for $\hat{a}$, $\hat{e}$, $\hat{s}$, and $\hat{p}$ can then be substituted into the right-hand sides of (1), and this will yield the equations for the nonangular elements α, e, s, and p in the form analogous to (5):

$$\frac{da}{dt} = L_a \sum_{\substack{k,l,m\\ i,j}} \{{}_aA_{2i,j}^{2k,2l,m} \sin[2k(\Omega' - \Omega) \pm 2l\omega' \pm 2i\omega \pm j\psi \pm mv'] +$$

$$+ {}_aB_{2i,j}^{1,2l,m} \cos(\Omega' - \Omega \pm 2l\omega' \pm 2i\omega \pm j\psi \pm mv')\},$$

$$\frac{de}{dt} = L_e \sum_{\substack{k,l,m\\ i,j}} \{{}_eA_{2i,j}^{2k,2l,m} \sin[2k(\Omega' - \Omega) \pm 2l\omega' \pm 2i\omega \pm$$

$$\pm j\psi \pm mv'] + {}_eB_{2i,j}^{1,2l,m} \cos(\Omega' - \Omega \pm 2l\omega' \pm 2i\omega \pm j\psi \pm mv')\}, \qquad (8)$$

$$\frac{ds}{dt} = L_s \sum_{\substack{k,l,m\\ i,j}} \{ {}_sA_{2i,j}^{2k,2l,m} \sin[2k(\Omega' - \Omega) \pm 2l\,\omega' \pm 2i\,\omega \pm j\,\psi \pm mv'] + {}_sB_{2i,j}^{1,2l,m} \cos(\Omega' - \Omega \pm 2l\,\omega' \pm 2i\,\omega \pm j\,\psi \pm mv')\} \tag{8}$$

and

$$\frac{dp}{dt} = L_p \sum_{\substack{k,l,m\\ i,j}} \{ {}_pA_{2i,j}^{2k,2l,m} \sin[2k(\Omega' - \Omega) \pm 2l\,\omega' \pm 2i\,\omega \pm j\,\psi \pm mv'] + {}_pB_{2i,j}^{1,2l,m} \cos(\Omega' - \Omega \pm 2l\,\omega' \pm 2i\,\omega \pm j\,\psi \pm mv')\},$$

where

$$k,\ l = 0,\ 1; \quad m = 0,\ 1,\ 2,\ 3,\ 4,\ 5;$$
$$i = 0,\ 1,\ 2; \quad j = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7;$$

and

$$L_a = \frac{a^2\sqrt{e^2-1}\cdot n}{64\,a'^3(1-e'^2)^3}\cdot\frac{2\hat{a}^3}{(1+\bar{e}\cos\psi)^2},$$
$$L_e = \frac{a^2\sqrt{e^2-1}\cdot n}{64\,a'^3(1-e'^2)^3}\cdot\frac{\hat{p}}{(1+\bar{e}\cos\psi)^2}, \tag{9}$$
$$L_s = -\frac{a^2\sqrt{e^2-1}\cdot n}{64\,a'^3(1-e'^2)^3}\cdot\frac{\hat{p}}{\hat{s}(1+\bar{e}\cos\psi)^2}$$

and

$$L_p = -\frac{a^2\sqrt{e^2-1}\cdot n}{64\,a'^3(1-e'^2)^3}\cdot\frac{2a^2(e^2-1)^2}{(1+\bar{e}\cos\psi)^2}.$$

For the coefficients A and B in front of the trigonometric functions inside the sums, we again write out only the coefficients in front of the constant terms in the sums:

$${}_aa_{20}^{000} = 6(3e'^2+2)(2-3s'^2)e^2s^2\left\{1-\frac{\varepsilon^2}{8}[38e^2-48-5s^2(11e^2-13)]\right\},$$
$${}_aa_{40}^{000} = -\frac{3}{8}\varepsilon^2(3e'^2+2)(2-3s'^2)e^4s^4,$$
$${}_aa_{\pm20}^{020} = \pm\frac{27}{2}e'^2s'^2e^2s^2\left\{1-\frac{\varepsilon^2}{8}[38e^2-48-5s^2(11e^2-13)]\right\},$$
$${}_aa_{\pm40}^{020} = \mp\frac{27}{32}\varepsilon^2e'^2s'^2e^4s^4,$$
$${}_aa_{00}^{200} = \frac{3}{4}\varepsilon^2(3e'^2+2)s'^2(13e^4-56e^2+8)\alpha s^2,$$
$${}_aa_{\pm20}^{200} = \mp3(3e'^2+2)s'^2e^2(1\mp\alpha)^2\times$$
$$\times\left\{1+\frac{\varepsilon^2}{8}[7(3e^2+1)\pm2\alpha(e^2+6)-5\alpha^2(11e^2-13)]\right\},$$
$${}_aa_{\pm40}^{200} = \pm\frac{3\varepsilon^2}{16}(3e'^2+2)s'^2e^4(1\mp\alpha)^2s^2,$$
$${}_aa_{00}^{2\pm20} = \frac{9\varepsilon^2}{16}e'^2(1\pm\alpha')^2(13e^4-56e^2+8)\alpha s^2,$$
$${}_aa_{\pm20}^{2\overset{+}{[-]}20} = \mp\frac{9}{4}e'^2(1\underset{[-]}{+}\alpha')^2e^2(1\mp\alpha)^2\times$$
$$\times\left\{1+\frac{\varepsilon^2}{8}[7(3e^2+1)\pm2\alpha(e^2+6)-5\alpha^2(11e^2-13)]\right\},$$
$${}_aa_{\pm40}^{2\overset{+}{[-]}20} = \pm\frac{9\varepsilon^2}{64}e'^2(1\underset{[-]}{+}\alpha')^2e^4(1\mp\alpha)^2; \tag{10}$$

$$ {}_aB_{00}^{100} = -3\varepsilon^2 \alpha' s' (3e'^2+2) s [8(e^2-1)^2 - 5 s^2 e^2 (e^2-8)], \tag{10} $$

$$ {}_aB_{\pm 20}^{100} = 24\alpha' s' (3e'^2+2)(1\mp\alpha) s e^2 \times $$

$$ \times \left\{ 1 + \frac{\varepsilon^2}{8} [18e^2 - 11 \pm \alpha(e^2+6) - 5\alpha^2(11e^2-13)] \right\}, $$

$$ {}_aB_{\pm 40}^{100} = -\frac{3}{2}\varepsilon^2 \alpha' s' (3e'^2+2)(1\mp\alpha) s^3 e^4, $$

$$ {}_aB_{00}^{1\pm 20} = \pm\frac{9}{4}\varepsilon^2 e'^2 (1\pm\alpha') s' s [8(e^2-1)^2 - 5 s^2 e^2 (e^2-8)], $$

$$ {}_aB_{\pm 20}^{1\underset{[-]}{+}20} = \underset{[+]}{-} 18 e'^2 s' (1 \underset{[-]}{+} \alpha') s (1\mp\alpha) e^2 \times $$

$$ \times \left\{ 1 + \frac{\varepsilon^2}{8} [18e^2 - 11 \pm \alpha(e^2+6) - 5\alpha^2(11e^2-13)] \right\}, $$

$$ {}_aB_{\pm 40}^{1\underset{[-]}{+}20} = \underset{[-]}{+} \frac{9}{8}\varepsilon^2 e'^2 s' (1 \underset{[-]}{+} \alpha') e^4 s^3 (1\mp\alpha); $$

$$ {}_ea_{20}^{000} = 2(3e'^2+2)(2-3s'^2) e^2 s^2 \left\{ 3 + \frac{\varepsilon^2}{8} [55(e^2-1) - 3\alpha^2(91e^2-69)] \right\}, $$

$$ {}_ea_{40}^{000} = -\frac{3}{8}\varepsilon^2 (3e'^2+2)(2-3s'^2) e^4 s^4, $$

$$ {}_ea_{\pm 20}^{020} = \pm\frac{9}{8} s'^2 e'^2 s^2 e^2 \left\{ 3 + \frac{\varepsilon^2}{8} [55(e^2-1) - 3\alpha^2(91e^2-69)] \right\}, $$

$$ {}_ea_{\pm 40}^{020} = \mp\frac{27}{32} \varepsilon^2 s'^2 e'^2 s^4 e^4, $$

$$ {}_ea_{00}^{200} = \frac{3}{4}\varepsilon^2 s'^2 (3e'^2+2) \alpha s^2 (57e^4 - 44e^2 - 48), $$

$$ {}_ea_{\pm 20}^{200} = \mp 3 s'^2 (3e'^2+2)(1\mp\alpha)^2 e^2 \times $$

$$ \times \left\{ 1 + \frac{\varepsilon^2}{8} [17e^2 + 11 \pm 2\alpha(e^2+6) - \alpha^2(91e^2-69)] \right\}, $$

$$ {}_ea_{\pm 40}^{200} = \pm\frac{3\varepsilon^2}{16} s'^2 (3e'^2+2) s^2 (1\mp\alpha)^2 e^4, $$

$$ {}_ea_{00}^{2\pm 20} = \frac{9\varepsilon^2}{16} (1\pm\alpha')^2 e'^2 \alpha s^2 (57e^4 - 44e^2 - 48), $$

$$ {}_ea_{\pm 20}^{2\underset{[-]}{+}20} = \mp\frac{9}{4} (1 \underset{[-]}{+} \alpha')^2 e'^2 (1\mp\alpha)^2 e^2 \times $$

$$ \times \left\{ 1 + \frac{\varepsilon^2}{8} [17e^2 + 11 \pm 2\alpha(e^2+6) - \alpha^2(91e^2-69)] \right\}, $$

$$ {}_ea_{\pm 40}^{2\underset{[-]}{+}20} = \pm\frac{9\varepsilon^2}{64} (1 \underset{[-]}{+} \alpha')^2 e'^2 (1\mp\alpha)^2 s^2 e^4; $$

$$ {}_eB_{00}^{100} = \frac{3}{2}\varepsilon^2 \alpha' s' (3e'^2+2) s [e^4 - 20e^2 - 16 - \alpha^2(57e^4 - 44e^2 - 48)], $$

$$ {}_eB_{\pm 20}^{100} = 24\alpha' s' (3e'^2+2)(1\mp\alpha) s e^2 \times $$

$$ \times \left\{ 1 + \frac{\varepsilon^2}{8} [18e^2 - 11 \pm \alpha(e^2+6) - \alpha^2(91e^2-69)] \right\}, $$

$$ {}_eB_{\pm 40}^{100} = -\frac{3}{4}\varepsilon^2 \alpha' s' (3e'^2+2)(1\mp\alpha) s^3 e^4, $$

$$ {}_eB_{00}^{1\pm 20} = \mp\frac{9}{8}\varepsilon^2 (1\pm\alpha') s' e'^2 s [e^4 - 20e^2 - 16 - \alpha^2(57e^4 - 44e^2 - 48)], $$

$$ {}_eB_{\pm 20}^{1\underset{[-]}{+}20} = \underset{[+]}{-} 9 (1 \underset{[-]}{+} \alpha') s' e'^2 (1\mp\alpha) s e^2 \times $$

$$ \times \left\{ 1 + \frac{\varepsilon^2}{8} [18e^2 - 11 \pm \alpha(e^2+6) - \alpha^2(91e^2-69)] \right\}, $$

$$ {}_eB_{\pm 40}^{1\underset{[-]}{+}20} = \underset{[-]}{+} \frac{9\varepsilon^2}{16} (1 \underset{[-]}{+} \alpha') s' e'^2 (1\mp\alpha) s^3 e^4; $$

$$ {}_sa_{20}^{000} = -\varepsilon^2 (3e'^2+2)(2-3s'^2) e^2 \alpha^2 s^2 (2-9s^2), $$

$$ {}_sa_{\pm 20}^{020} = \mp\frac{9}{4} \varepsilon^2 e'^2 s'^2 e^2 \alpha s^2 (2-9s^2), $$

$$
\begin{aligned}
{}_s a_{00}^{200} &= -12 s'^2 (3e'^2 + 2)\,\alpha\,\widehat{s^2}\left\{1 + \frac{\varepsilon^2}{8}\left[17e^2 + 20 - \alpha^2 (73e^2 + 16)\right]\right\}, \\
{}_s a_{\pm 20}^{200} &= \frac{3}{4}\varepsilon^2 s'^2 (3e'^2 + 2)\,\alpha (1 \mp \alpha) s^2 (5 \mp 5\alpha + 6\alpha^2) e^2, \\
{}_s a_{00}^{2\underset{[-]}{+}20} &= -\frac{9}{4} e'^2 (1 \pm \alpha')^2 \alpha\,\widehat{s^2} \times \\
&\times \left\{1 + \frac{\varepsilon^2}{8}\left[17e^2 + 20 - \alpha^2 (73e^2 + 16)\right]\right\}, \\
{}_s a_{\pm 20}^{2\underset{[-]}{+}20} &= \frac{9\varepsilon^2}{16} e'^2 (1 \underset{[-]}{+} \alpha')^2 e^2 \alpha s^2 (1 \mp \alpha)(5 \mp 5\alpha + 6\alpha^2); \\
{}_s B_{00}^{100} &= 24\alpha' s' (3e'^2 + 2)\,\alpha^2 \widehat{s}\left\{1 + \frac{\varepsilon^2}{8}\left[15e^2 + 16 - \alpha^2 (77e^2 + 12)\right]\right\}, \\
{}_s B_{\pm 20}^{100} &= \mp\frac{3}{2}\varepsilon^2 \alpha' s' (3e'^2 + 2)\,\alpha\widehat{s}(1 \mp \alpha)(3 \mp 15\alpha + \alpha^2 \pm 20\alpha^3) e^2, \\
{}_s B_{00}^{1\pm 20} &= \pm 18 e'^2 (1 \pm \alpha') s' \alpha^2 \widehat{s}\left\{1 + \frac{\varepsilon^2}{8}\left[15e^2 + 16 - \alpha^2 (77e^2 + 12)\right]\right\}, \\
{}_s B_{\pm 20}^{1\underset{[-]}{+}20} &= \underset{[\mp]}{\pm}\frac{9}{8}\varepsilon^2 e'^2 s' (1 \underset{[-]}{+} \alpha') e^2 s \alpha (1 \mp \alpha)(3 \mp 15\alpha + \alpha^2 \pm 20\alpha^3); \\
{}_p a_{20}^{000} &= \varepsilon^2 (3e'^2 + 2)(2 - 3s'^2) e^2 s^2 (2 - 3s^2), \\
{}_p a_{\pm 20}^{020} &= \pm\frac{9}{4}\varepsilon^2 e'^2 s'^2 e^2 s^2 (2 - 3s^2), \\
{}_p a_{00}^{200} &= -3\varepsilon^2 (3e'^2 + 2) s'^2 (11e^2 + 14) s^2 \alpha, \\
{}_p a_{\pm 20}^{200} &= \mp\frac{3}{2}\varepsilon^2 (3\varepsilon'^2 + 2) s'^2 e^2 (1 \mp \alpha)^2 (1 + \alpha^2), \\
{}_p a_{00}^{2\pm 20} &= -\frac{9}{4}\varepsilon^2 e'^2 (1 \pm \alpha')^2 (11e^2 + 14) s^2 \alpha, \\
{}_p a_{\pm 20}^{2\underset{[-]}{+}20} &= \mp\frac{9}{8} e^2 e'^2 (1 \underset{[-]}{+} \alpha')^2 e^2 (1 \mp \alpha)^2 (1 + \alpha^2); \\
{}_p B_{00}^{100} &= 6\varepsilon^2 \alpha' s' (3e'^2 + 2) s\left[10e^2 + 12 - s^2 (11e^2 + 14)\right], \\
{}_p B_{\pm 20}^{100} &= 6\varepsilon^2 \alpha' s' (3e'^2 + 2) s (1 \mp \alpha)\alpha^2 e^2, \\
{}_p B_{00}^{1\pm 20} &= \mp\frac{9}{2}\varepsilon^2 e'^2 (1 \pm \alpha') s' s\left[10e^2 + 12 - s^2 (11e^2 + 14)\right], \\
{}_p B_{\pm 20}^{1\underset{[-]}{+}20} &= \underset{[+]}{-}\frac{9}{2}\varepsilon^2 e'^2 (1 \underset{[-]}{+} \alpha') s' e^2 (1 \mp \alpha)\alpha^2 s.
\end{aligned}
\qquad (10)
$$

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