A NEW VERSION OF NONLINEAR ELECTRODYNAMICS (II)

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Vol. 33, No. 6, pp. 3-8, 1978

UDC 530.12

We obtain the dynamical (wave and soliton) solutions of the equations of a nonlinear electrodynamics, the nonlinearity of which is caused by the contortion of space-time.

In part (I) of this paper we derived the equations of a nonlinear electromagnetic field (Eq. (6) of Ref. [1]), the nonlinearity of which was due to the spin-spin interaction induced by the contortion of space-time, and also studied the electrostatic solutions of these equations both in Minkowski space and in a curved space-time.

In this part we will examine the dynamical solutions of Eq. (6) of Ref. [1]*

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\alpha}}\left(\sqrt{-g}\,F^{\alpha\beta}\right)=\frac{4\pi}{c}J^{\beta},$$

where

$$F_{\mu\nu}=\tilde{F}_{\mu\nu}+\frac{2bA_{[\mu}\tilde{F}_{\nu]\gamma}A^{\gamma}}{1+bA^{2}};\quad J^{\beta}=-\frac{bc}{4\pi}F^{\alpha\beta}F_{\alpha\gamma}A^{\gamma}.$$

3. **Wave solution.** We will examine first the wave solution for the particular case that the 4-potential of the problem has the form $A_{\mu}=(0, A(x^0, y), 0, 0)$, which corresponds to a plane-polarized wave propagating along the y axis. In this case the system of equations (6) of Ref. [1] reduces to the single equation

$$\left(\frac{\partial^2 A}{\partial x^{02}}-\frac{\partial^2 A}{\partial y^2}\right)+\frac{b}{1-bA^2}A\left[\left(\frac{\partial A}{\partial x^0}\right)^2-\left(\frac{\partial A}{\partial y}\right)^2\right]=0.$$

In the new variables $\xi=x^0+y$, $\eta=x^0-y$ this equation is written as

$$\frac{\partial A}{\partial\xi\,\partial\eta}+\frac{b}{1-bA^2}A\frac{\partial A}{\partial\xi}\frac{\partial A}{\partial\eta}=0. \tag{1}$$

The solution of Eq. (1) has the form

$$A=\begin{cases}\frac{1}{\sqrt{b}}\sin\left[f_1(\xi)+f_2(\eta)\right] & A^2<\frac{1}{b}, \quad (2a)\\ \frac{1}{\sqrt{b}}\,\mathrm{ch}\left[\psi_1(\xi)+\psi_2(\eta)\right] & A^2>\frac{1}{b}. \quad (2b)\end{cases}$$

Here $f_1(\xi)$, $f_2(\eta)$, $\psi_1(\xi)$, and $\psi_2(\eta)$ are arbitrary functions.

To examine the interaction of these plane waves we will start from Eq. (2a)**, writing this solution in the more convenient form:

$$A=\frac{1}{\sqrt{b}}\left(\varphi_1(\xi)\sqrt{1-\varphi_2^2(\eta)}+\varphi_2(\eta)\sqrt{1-\varphi_1^2(\xi)}\right). \tag{3}$$

*All of the notation was introduced in Ref. [1]; the equations are examined in Minkowski space.

**Note that the solitary plane electromagnetic wave does not interact with the contortion field proper.

where $\varphi_{1,2}=\sin f_{1,2}$ and $|\varphi_{1,2}|<1$.

We will assume [2, 3] that for $t \to -\infty$ our solution can be written in the form of a superposition of two waves. It is readily seen that this solution satisfies such an initial condition if one chooses plane waves concentrated in a limited space-time region. In moving toward each other, these waves are altered and will undergo interaction, during which they will deform. To obtain the result of this interaction, one needs only examine the solution (3) at $t \to +\infty$. It turns out that solution (3) again goes over to the superposition of undeformed waves. Thus the interaction does not ultimately lead to alteration of the scattering waves.

We point out in conclusion that for a single plane wave all the components of the current J^{β} introduced in Ref. [1] are equal to zero, i.e., an electromagnetic wave does not carry charge. The components of the current will differ from zero only in the interaction region. One can see this by writing our the explicit form of the nonzero components of the current for solution (2a)

$$J_1=\frac{c}{\pi\sqrt{b}}\,\frac{\partial f_1}{\partial\xi}\,\frac{\partial f_2}{\partial\eta}\,\frac{\sin(f_1+f_2)}{\cos^2(f_1+f_2)}.$$

4. Soliton solution. There has been much attention paid lately to the investigation of the special solutions of nonlinear equations which correspond to moving particle-like objects—solitons [4]. It is of interest in this regard to obtain similar nonstationary solutions for Eq. (6) of Ref. [1]. We seek a solution for this system of equations of the form

$$A_\mu=(A_0(x^0,x,\rho),\ A_1(x^0,x,\rho),0,0),$$

where

$$\rho=\sqrt{y^2+z^2}.$$

Setting equal to zero the components of the current J_y and J_z

$$J_y=J_z=-\frac{b\widetilde{F}_{10}}{(1+bA^2)}\left(\frac{\partial A_0}{\partial\rho}A_1-\frac{\partial A_1}{\partial\rho}A_0\right),$$

which amounts to setting to zero the expression $\frac{\partial A_0}{\partial\rho}A_1-\frac{\partial A_1}{\partial\rho}A_0$, we obtain

$$A_1(x^0,x,\rho)=f(x^0,x)A_0(x^0,x,\rho).$$

Moreover, if it is assumed that $f(x^0,x)=-\beta=\mathrm{const}$ (this is equivalent, as we shall see, to examining the one-particle soliton solutions), it will follow from Eq. (6) of Ref. [1] that $A_0(x^0,x,\rho)=A_0(x-\beta x^0,\rho)$. Note that the constant β characterizes the propagation velocity of the object described by A_0. By virtue of the above assumption, the system of equations (6) of Ref. [1] reduces to the single equation in the variables $\xi=x-\beta x^0$, and ρ

$$\frac{\partial}{\partial\xi}\left(\frac{\widetilde{F}_{10}}{1+bA^2}\right)+\frac{1}{\rho}\,\frac{\frac{\partial A_0}{\partial\rho}}{1+bA^2}+\frac{\partial}{\partial\rho}\left(\frac{\frac{\partial A_0}{\partial\rho}}{1+bA^2}\right)=-\frac{bA_0}{(1+bA^2)^2}\left\{\widetilde{F}_{10}^2+\left(\frac{\partial A_0}{\partial\rho}\right)^2(1-\beta^2)\right\},\qquad(4)$$

where $\widetilde{F}_{10}=(1-\beta^2)\partial_\xi A_0;\ A^2=(1-\beta^2)A_0^2$.

Let $|\beta|<1\ (\beta\neq0)$. Then, making the change of variable $\eta=\frac{\xi}{\sqrt{1-\beta^2}}$, we rewrite Eq. (4) in the form

$$\frac{\partial^2u}{\partial\eta^2}+\frac{\partial^2u}{\partial\rho^2}+\frac{1}{\rho}\,\frac{\partial u}{\partial\rho}=\frac{u}{1+u^2}\left[\left(\frac{\partial u}{\partial\eta}\right)^2+\left(\frac{\partial u}{\partial\rho}\right)^2\right]$$

where $u=\sqrt{b}\,\sqrt{1-\beta^2}\,A_0$.

This equation, which is nonlinear in u, reduces to a linear equation in the new variable $\varphi = \mathrm{arcsh}\, u$

$$\frac{\partial^2\varphi}{\partial\eta^2} + \frac{\partial^2\varphi}{\partial\rho^2} + \frac{1}{\rho}\,\frac{\partial\varphi}{\partial\rho} = 0.$$

We finally have

$$A_0 = \frac{1}{\sqrt{b}\,\sqrt{1-\beta^2}}\,\mathrm{sh}\,\varphi(x-\beta x^0, \rho); \quad A_1 = -\beta A_0,$$

$$\varphi(x-\beta x^0, \rho) = \int_{-\infty}^{+\infty} \varphi(\lambda, x-\beta x^0, \rho)\, d\lambda,$$

where

$$\varphi(\lambda, \eta, \rho) = \begin{cases} A_1(\lambda)\, e^{i\sqrt{\lambda}\eta} I_0(\sqrt{\lambda}\rho) + B_1(\lambda)\, e^{i\sqrt{\lambda}\eta} K_0(\sqrt{\lambda}\rho) + \\ + C_1(\lambda)\, e^{-i\sqrt{\lambda}\eta} I_0(\sqrt{\lambda}\rho) + D_1(\lambda)\, e^{-i\sqrt{\lambda}\eta} K_0(\sqrt{\lambda}\rho) & \lambda > 0 \\ (A_3\eta + B_3)\ln\frac{\rho}{\rho_0} & \lambda = 0 \\ A_2(\lambda)\, e^{\sqrt{-\lambda}\eta} J_0(\sqrt{-\lambda}\rho) + B_2(\lambda)\, e^{\sqrt{-\lambda}\eta} N_0(\sqrt{-\lambda}\rho) + \\ + C_2(\lambda)\, e^{-\sqrt{-\lambda}\eta} J_0(\sqrt{-\lambda}\rho) + D_2(\lambda)\, e^{-\sqrt{-\lambda}\eta} N_0(\overline{-\lambda}\rho) & \lambda < 0 \end{cases}$$

$A_{1,2}(\lambda)$, $B_{1,2}(\lambda)$, $C_{1,2}(\lambda)$, and $D_{12}(\lambda)$ are arbitrary functions of λ; A_3, B_3 and ρ_0 are constants of integration; $J_0(z)$, $I_0(z)$, $K_0(z)$, and $N_0(z)$ are Bessel functions; and λ is a continuous parameter.

We will investigate the particular solution of Eq. (4) (which can also be obtained from the static solution (11) of Ref. [1] by means of a Lorentz transformation):

$$A_0 = \frac{1}{\sqrt{b}\,\sqrt{1-\beta^2}}\,\mathrm{sh}\,\frac{\sqrt{b}\,q}{\sqrt{\left(\frac{x-\beta x^0}{\sqrt{1-\beta^2}}\right)^2 + \rho^2}}; \quad A_1 = -\beta A_0. \qquad (5)$$

The electric and magnetic field strengths E and H which correspond to solution (5) describe the soliton

$$E_x = F_{01} = \frac{q}{\sqrt{1-\beta^2}}\,\frac{1}{\mathrm{ch}\,\varphi}\,\frac{x-\beta x^0}{r_1^3}, \quad H_x = F_{23} = 0;$$

$$E_y = F_{02} = \frac{q}{\sqrt{1-\beta^2}}\,\frac{1}{\mathrm{ch}\,\varphi}\,\frac{y}{r_1^3}, \quad H_y = F_{31} = \beta E_z;$$

$$E_z = F_{03} = \frac{q}{\sqrt{1-\beta^2}}\,\frac{1}{\mathrm{ch}\,\varphi}\,\frac{z}{r_1^3}, \quad H_z = F_{12} = -\beta E_y;$$

$$|\mathbf{E}| = \frac{q}{\sqrt{1-\beta^2}}\,\frac{1}{\mathrm{ch}\,\varphi}\,\frac{((x-\beta x^0)^2+\rho^2)^{1/2}}{r_1^3},$$

$$|\mathbf{H}| = \frac{q\beta}{\sqrt{1-\beta^2}}\,\frac{\rho}{\mathrm{ch}\,\varphi\, r_1^3}, \quad (\mathbf{EH}) = 0, \; J_1 = -\beta c\rho^*; \; J_2 = J_3 = 0;$$

$$\rho^* = \frac{\sqrt{b}\,q^2}{4\pi}\,\frac{\mathrm{sh}\,\varphi}{\sqrt{1-\beta^2}\,\mathrm{ch}^2\varphi\, r_1^4}.$$

Here $\varphi = \frac{\sqrt{b}\,q}{r_1}$, $r_1 = \sqrt{\left(\frac{x-\beta x^0}{\sqrt{1-\beta^2}}\right)^2 + \rho^2}$, and $q = \int \rho^* dv$ is the charge of the soliton.

The qualitative behavior of the electric field as a function of the coordinates and velocity is shown in Figs. 1 through 4.

We see that at the center of the soliton $x = \beta x_0$, $y = z = 0$ the electromagnetic field is equal to zero. Notice the interesting behavior of the electric and magnetic field strengths as functions of the velocity for $\beta \to 1$

$$E_\parallel = E_x \to 0$$

$$\begin{cases} E_\perp = \sqrt{E_y^2 + E_z^2} \to 0 & x \neq \beta x^0 \\ E_\perp \to \infty & x = \beta x^0. \end{cases}$$

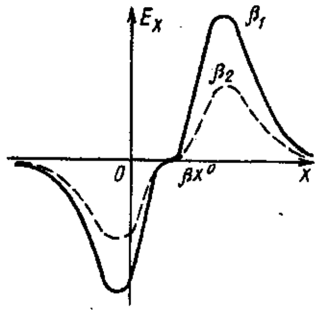

Fig. 1

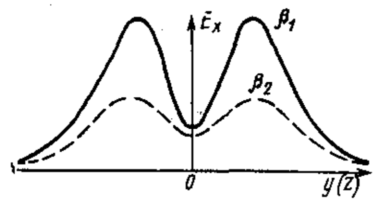

Fig. 2

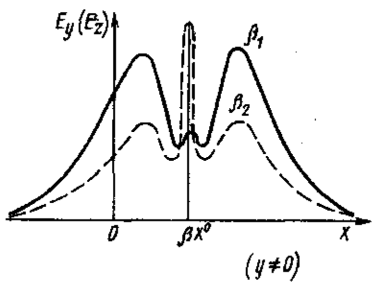

Fig. 3

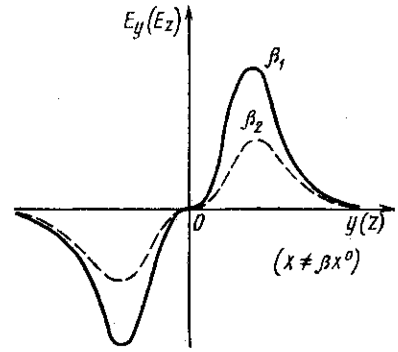

Fig. 4

The soliton energy, determined by the symmetric or canonical energy-momentum tensor (Eq. (12) of Ref. [1]), turns out as before to be divergent due to the self-action term. However, in the theory there is a finite quantity which is conserved with time; this can be termed the mass of the soliton. We will examine the tensor $t^{\alpha\beta}$, which satisfies the conservation law $t^{\alpha\beta}{}_{,\beta}=0$, as is easily seen if $t^{\alpha\beta}$ is written in the form (using Eq. (6) of Ref. [1])

$$t^{\alpha\beta} = -\frac{1}{4\pi}(F^{\nu\beta}A^{\alpha})_{,\nu}.$$

We will determine the energy and momentum of the soliton through the tensor $t^{\alpha\beta}$, which is the difference of the symmetric and canonical energy-momentum tensors. For the static solution (11) of Ref. [1] we obtain

$$m_0 = \frac{W_0}{c^2} = \frac{|q|}{\sqrt{b}\, c^2} = \frac{|q|}{\sqrt{k}}.$$

For the case of a moving soliton (5)

$$m = \frac{m_0}{\sqrt{1-\beta^2}}; \quad P_i = \left\{\frac{m_0 v}{\sqrt{1-\beta^2}}, 0, 0\right\}.$$

Thus one automatically obtains the correct relativistic connection between the electromagnetic energies of the moving and stationary charged particles and avoids the "paradox" of the classical electron [5].

And so in this paper we have constructed a new version of nonlinear electrodynamics. Its characteristic feature is its taking into account the action of the spin of the electromagnetic field on the contortion of the space-time continuum, the reaction of which leads to nonlinear terms in the equations of the theory which break their gauge invariance; this is interpreted as the occurrence of intrinsic sources of the electromagnetic field.

We have obtained in this paper a number of exact electrostatic particle-like solutions with field functions which are nonsingular everywhere in space and whose asymptotics at infinity are Coulombic. In this case the energy of the nonlinear electromagnetic field turns out to be finite for a rather broad range of seeding masses.

It was shown that the Proca field of the wave type is described by the sine-Gordon equation.

It is important that within the framework of the proposed theory there are exact three-dimensional solutions describing solitary waves, or solitons.

The detailed analysis of these soliton solutions for stability and the study of the quantum effects of the theory will be the subjects of further investigation by the authors.

The authors express profound gratitude to N. N. Kolesnikov, V. R. Khalilov, and the participants in the seminar conducted by Prof. D. D. Ivanenko for discussion of the results of this work.

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