CORRECT USE OF THE PALATINI PRINCIPLE IN GRAVITATIONAL THEORIES

V. N. Ponomarev and A. A. Tseitlin

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It is shown that the correct variational problem in metric theory of gravitations, based on independent variation with respect to the coefficient of affine connectivity, is equivalent to the problem of finding the conditional extremum of the functional of the action

$$S=\int\limits_{\Omega}\sqrt{-g}\,R d^4x.$$

The gist of the variational principle proposed by Palatini [1] in order to construct a canonical scheme of the Einstein theory of gravitation, is that the metric-tensor components g_{ik} and the affine connectivity components Γ^i_{jk} are treated independently in the course of variation of the action

$$S=\int\limits_{\Omega}\sqrt{-g}\,L d^4x=\int\limits_{\Omega}(R-2\varkappa L_m)\sqrt{-g}\,d^4x$$

(R is the scalar curvature, $\varkappa$ is the Einstein gravitational constant, $L_m=L_m(\Psi_A,\ \nabla_i\Psi_A,\ g_{ik})$ is the Lagrangian of the matter, Ψ_A is a function of the field, A is a generalized index, and ∇_i is the covariant derivative with respect to Γ^i_{jk}). A justification for this principle must apparently be sought in the independence of the metric structure and the structure of affine connectivity on a manifold.

If the Lagrangian of the matter does not depend on the connectivity, and the connectivity is symmetrical with respect to the lower indices (torsion $Q^i_{jk}\equiv 2\Gamma^i_{[jk]}=0$), we obtain as a result of varying the action with respect to connectivity, besides field equations that coincide with the Einstein equations, also the metric condition $\tilde{\nabla}_i g_{jk}=0$ ($\tilde{\nabla}_i$ is the covariant derivative with respect to $\Gamma^i_{jk}=\Gamma^i_{(jk)}$), whence

$$\Gamma^i_{jk}=\{^i{}_{jk}\}=\frac{1}{2}g^{il}(g_{jl,k}+g_{kl,j}-g_{jk,l}).$$

The Palatini variational principle is the basis of various generalizations and alternative formulations of the general theory of relativity [2-5].

At the same time, it is far from always used consistently. When one deals with more general manifolds compared with the Riemannian ones, then the variation procedure, according to Palatini, can be realized in two manners:

1. We can forego a priori limitations on the connectivity and Lagrangian of the matter, vary the Lagrangian L as before, assuming g_{ik} and Γ^i_{jk} to be independent. A consistent application of this procedure leads to the Weyl spaces.

2. By starting from physical or geometrical premises, one can impose limitations on the objects of the theory and regard these limitations as constraints that must be added to L with indeterminate Lagrangian multipliers. In this case the problem of obtaining the field equations becomes a variational problem for finding the conditional extremum of the Lagrange function.

In particular, in the Einstein-Cartan theory (ECT) [2], the field equation of which is derived on the basis of the Palatini principle, the connectivity satisfies the metricity condition, but this is not taken to account at all in the variation procedure, a fact which is incorrect from our point of view. In connection with the foregoing, it would be correct to take into account in the Lagrangian from the very beginning the metricity condition $\nabla_i g_{jk}=0$, i.e., to obtain the field equations from the Lagrangian

$$\sqrt{-g}\,L=\sqrt{-g}\,(L+\Lambda^{ijk}\nabla_i g_{jk}) \qquad (1)$$

(where $\Lambda^{ijk}(x^l)$ are indeterminate Lagrangian multipliers). By varying the action with the Lagrangian (1) with respect to g_{ij}, Γ^i_{jk}, Λ^{ijk} and Ψ_A, we obtain the following field equations:

$$R^n{}_{(i|n|k)}-\frac{1}{2}g_{ik}R^{ln}{}_{ln}+(\nabla_j-Q^l{}_{jl})\Lambda^j{}_{ik}=\varkappa T_{ik}; \qquad (2)$$

$$Q^i{}_{jk}+\delta^i_j Q^l{}_{kl}-\delta^i_k Q^l{}_{jl}=\varkappa cS^i{}_{jk}, \qquad (3)$$

$$\nabla_i g_{jk}=0,\ \Lambda^j{}_{ik}=-\varkappa P^j{}_{ik};$$

$$\frac{\delta L_m}{\delta\Psi_A}\equiv\frac{\partial L_m}{\partial\Psi_A}-(\nabla_i-Q^l{}_{il})\frac{\partial L_m}{\partial\nabla_i\Psi_A}=0, \qquad (4)$$

where $T_{ik}=\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\,L_m)}{\partial g^{ik}}$ is the metric energy-momentum tensor; $cS^i{}_{jk}=2\frac{\partial L_m}{\partial\Gamma^n{}_{il}}\delta^n_{[j}g_{k]l}$ (c is the speed of light);

$$P^i{}_{jk}=\frac{\partial L_m}{\partial\Gamma^n{}_{il}}\delta^n_{(j}g_{k)l}.$$

Eliminating $\Lambda^j{}_{ik}$ from (2), we get

$$R^l{}_{(i|l|k)}-\frac{1}{2}g_{ik}R^{ln}{}_{ln}=\varkappa[T_{ik}+(\nabla_j-Q^l{}_{jl})P^j{}_{(ik)}]. \qquad (5)$$

In vacuum, Eqs. (5), just as in the usual Einstein-Cartan theory, reduce to the Einstein equations, while in the presence of matter they differ from the terms $(\nabla_j-Q^l{}_{jl})P^j{}_{(ik}$ corresponding to the ECT equations in the energy-momentum tensor. As a result of this increment, the right-hand side of (5) will contain the symmetrical part of the canonical energy-momentum tensor

$$t^i{}_k=\frac{\partial L_m}{\partial\nabla_i\Psi_A}\nabla_k\Psi_A-\delta^i{}_k L_m.$$

Indeed, consider the infinitesimal transformation $\bar{x}^i=x^i+\xi^i\delta t$, where $\xi^i(x^k)$ is an arbitrary smooth vector field and δt is an infinitesimally small parameter.

The Lie derivative of $\sqrt{-g}\,L_m(\Psi_A,\nabla_i\Psi_A,g_{ik})$ along the vector field ξ^i, which induces this transformation is of the form

$$L_\xi(\sqrt{-g}\,L_m)=\frac{\partial(\sqrt{-g}\,L_m)}{\partial\Psi_A}L_\xi\Psi_A+\frac{\partial(\sqrt{-g}\,L_m)}{\partial\nabla_i\Psi_A}L_\xi\nabla_i\Psi_A+\frac{\partial(\sqrt{-g}\,L_m)}{\partial g_{ik}}L_\xi g_{ik}, \qquad (6)$$

where

$$L_\xi\Psi_A=\nabla_i\Psi_A\xi^i-\Omega^{kB}_{nA}\Psi_B(\nabla_k\xi^n-Q^n{}_{jk}\xi^j),$$

$$L_\xi\nabla_i\Psi_A=\nabla_k\nabla_i\Psi_A\xi^k+(\Omega^k{}_n{}^B{}_A\delta^l{}_i+\delta^k{}_i\delta^l{}_n\delta^B{}_A)\nabla_l\Psi_B(\nabla_k\xi^n-Q^n{}_{jk}\xi^j),$$

$$L_\xi g_{ik}=2\nabla_{(i}\xi_{k)}+2Q_{(ik)j}\xi^j,$$

$$\nabla_i\Psi_A=\Psi_{A,i}-\Omega^{nB}_{kA}\Psi_B\Gamma^k_{ni}.$$

On the other hand, since $\sqrt{-g}\,L_m$ is a scalar density of the weight +1, we have

$$L_\xi(\sqrt{-g}\,L_m)=\nabla_i(\sqrt{-g}\,L_m)\xi^i+(\nabla_i\xi^i-Q^i{}_{ji}\xi^j)\sqrt{-g}\,L_m. \qquad (7)$$

Comparing (6) and (7), we obtain

$$\frac{\partial L_m}{\partial\Psi_A}\Omega^{iB}_{kA}\Psi_B+t^i{}_k-T^i{}_k+(\nabla_a-Q^l{}_{al})\Lambda^{ai}{}_k=0, \qquad (8)$$

$$(\nabla_i - Q^l{}_{il})\, t^i{}_j - Q^k{}_{ij}\, t^i{}_k - CS^i{}_{kl} R^{kl}{}_{ij} = 0, \tag{9}$$

where

$$N^{ai}{}_k \equiv \frac{\partial L_m}{\partial \nabla_a \Psi_A} \Omega^{iB}_{kA} \Psi_B = - \frac{\partial L_m}{\partial \nabla_l \Psi_A} \frac{\partial \nabla_l \Psi_A}{\partial \Gamma^k{}_{ia}} = - \frac{\partial L_m}{\partial \Gamma^k{}_{ia}}.$$

From the definition for $N^{\alpha j}_k$ it follows that $P^j{}_{ik} = -N^j{}_{(ik)}$, whereas $cS^i{}_{jk} = 2N^i{}_{[jk]}$, i.e., S^i_{jk} coincides with the canonical spin density tensor [7]. By virtue of the field equations (4) we obtain from (8)

$$\begin{aligned} t_{(ik)} &= T_{ik} + (\nabla_j - Q^l{}_{jl})\, P^j{}_{(ik)}, \\ 2ct_{[ik]} &= (\nabla_a - Q^l{}_{al})\, S^a{}_{ki}. \end{aligned} \tag{10}$$

Thus, within the framework of the Einstein-Cartan gravitational theory, matter is described by a canonical energy-momentum tensor t_{ik} and a spin tensor S^i_{jk}. (A similar version of the present article was submitted to the collection "Problems of Theory of Gravitation and Elementary Particles" in March 1976, at the Fourth Soviet Gravitational Conference (Minsk, 1976). Professor S. Bazanski pointed out to the authors a paper by W. Kopczynski (Bull. Pol. Acad. Sci. (ser. mat. astr.), vol. 23, p. 467, 1976), who arrived at the same conclusions.)

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Department of Theoretical Physics