SPECTRUM OF PHOTONS EQUIVALENT TO FIELD OF PARTICLE MOVING IN EXTERNAL FIELD

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An expression is obtained for the spectrum of photons equivalent to a particle moving in an external field; this is a generalization of the well-known result of Weizsacker and Williams.

Inelastic processes play a decisive role in high-energy physics. However, the theoretical description of these processes encounters serious mathematical difficulties. For this reason it is of interest to explore approximate methods which permit a rather simple study of the various characteristics in some particular regions of the phase space of the final particles. The Weizsacker-Williams method and its modifications have been successfully applied in a number of papers [1-6].

At the present time, there is increasing interest in the study of inelastic processes in the presence of external fields. The effect of the field will be noticeable when the formation length of the process is not shorter than the length at which the field deviates the particles by appreciable angles. For evaluating the effects of first and higher orders in the fine-structure constant, the crossed-fields model has turned out to be especially promising, since, as was shown by Nikishov and Ritus [7], the results of the calculations, when written in invariant form, are valid for a constant field $F_{\mu\nu}$ of arbitrary form. In the present paper, we use the method of Ref. [7] to generalize the equivalent-photons approximation for calculating processes occurring in a constant external field.

We will examine the collision of particles having four-momenta p_1 and p_2 and charges e and Ze in a crossed field whose vector potential is $A_\mu = a_\mu \varphi$, $\varphi = kx$. We will assume that particle p_2 is spinless and has charge Ze, and in the rest frame $p_1 = 0$, $\varepsilon_2 \gg m_2$. As a result of the collision, a group of particles with total momentum p' is produced, and particle p_2^1 acquires momentum p_2'. This process has an S-matrix element:

$$S = (2\pi)^4 \int ds\, ds_1 \frac{4\pi Ze^2}{q^2(s_1)} j_\mu(s_1) J^\mu(s - s_1),$$

$$\delta^{(4)}(p_1 + p_2 + sk - p' - p_2') \frac{1}{\sqrt{2\varepsilon_1 2\varepsilon_2 2\varepsilon_2' \ldots}}. \quad (1)$$

Here $q(s_1) = p' - p - s_1 k$, j_μ is the current of process $p_1 + q = p'$ with virtual photon q_μ, and J_μ is the Fourier transform of the vertex current of particle p_2:

$$J_\mu(s) = \frac{1}{2\pi}\int \overline{\psi}_{p_2'}\overleftrightarrow{\pi}\,\psi_{p_2} e^{-i(p_2'-p_2)x+is\varphi}\,d\varphi, \tag{2}$$

where ψ_p is the Volkov function [7],

$$\psi_p = \frac{1}{\sqrt{2\varepsilon}}e^{iS_p},\quad S_p = -px + W,\ W = \frac{1}{2kp}\int[(p-eA)^2 - m^2]\,d\varphi,$$

which satisfies the Klein-Gordon equation.

The cross section of the process is

$$d\sigma = (2\pi)^4\delta^{(4)}(p_1 + p_2 + sk - p' - p_2')\frac{(4\pi Ze^2)^2}{4I}\cdot\frac{j_\mu(s_1)}{q^2(s_1)}\times$$
$$\times\frac{j_\nu^*(s_1')}{q^2(s_1')}\cdot J^\mu(s-s_1)J^{\nu*}(s-s_1')\,ds\,ds_1\,ds_1'\,\frac{dp_2'}{2\varepsilon_2'(2\pi)^3}\,\frac{d\rho'}{\delta(0)}, \tag{3}$$

where $d\rho' = \Pi\frac{dp'}{2\varepsilon'(2\pi)^3}$ is the statistical weight of the final particles,

$$I^2 = (p_1p_2)^2 - m_1^2m_2^2.$$

The current J_μ can be written in simpler form if it is represented as

$$J_\mu(s) = \frac{1}{2\pi}\int(-\partial_\mu S_{p_2} - \partial_\mu S_{p_2'} - 2eA_\mu)e^{i\Phi}\,d\varphi,$$
$$\Phi = s\varphi + W_{p_2} - W_{p_2'} \tag{4}$$

and the following relation is used:

$$\int_{-\infty}^{\infty}\frac{d}{d\varphi}e^{i\Phi}\,d\varphi = 0. \tag{5}$$

Taking Eq. (5) and the conservation law (1) into account, we find

$$J_\mu(s-s_1) = \frac{1}{\pi}\int\left(-\partial_\mu S_{p_2} - eA_\mu - \frac{q_\mu}{2}\right)e^{i\Phi}\,d\varphi. \tag{6}$$

Further, it is convenient to write the Fourier transform of the current J_μ in the form of an integral over the classical trajectory. To do this, we use the complete integral S of the Hamilton-Jacobi equation and write the law of motion $x_\mu(\tau)$ and four-momentum $mu_\mu(\tau)$ in parametric form (φ is the parameter):

$$-\frac{\partial S}{\partial p^\mu} = x_{0\mu},\ mu_\mu = -\partial_\mu S - eA_\mu. \tag{7}$$

The parameter φ is related to the proper time by $m_\varphi = kp\tau$. Taking Eq. (7) into account, one can show that the phase in Eq. (6) is

$$\Phi = \frac{kp_2}{kp_2'}\left(qx(\tau) - \frac{qp_2}{m_2}\tau\right) + (s-s_1)\frac{kp_2}{m_2}\tau.$$

Now the vertex account can be represented as

$$J_\mu(s) = \frac{1}{2\pi}\int[2m_2u - q]_\mu\exp\left[\frac{ikp_2}{kp_2'}\left(qx - \frac{qp_2}{m_2}\tau\right) + \frac{iskp_2}{m_2}\tau\right]\frac{dkp_2\tau}{m_2}. \tag{8}$$

This expression is of manifestly gradient-invariant form and is exact in the case of a particle moving in a crossed field. According to the ideas of Ref. [7], one can use the current in this form to examine processes in any homogeneous field. To do this, it is necessary that

$$\chi = \frac{e}{m^3}\sqrt{(F^{\mu\nu}p_\nu)^2}, \quad f = \frac{fe^2}{2m^2}(F_{\mu\nu})^2 \ll 1.$$

For a homogeneous magnetic field, the factor $\frac{kp_2}{kp_2'} = \frac{\varepsilon_2}{\varepsilon_2 - q_0}$ was obtained in [4] by "untangling" the exponential operator expressions.

Further, taking Eq. (8) into account, we find

$$J_\mu(s - s_1) J_\nu^*(s - s_1') = \delta(s_1 - s_1')\Gamma_{\mu\nu},$$

$$\Gamma_{\mu\nu} = \frac{1}{(2\pi)}\int\left[2mu\left(-\frac{\varphi}{2}\right) - q\right]_\mu \left[2mu\left(\frac{\varphi}{2}\right) - q\right]_\nu \times$$
$$\times \exp\left\{-\frac{ikp_2}{kp_2'}\left[qx\left(-\frac{\varphi}{2}\right) - qx\left(\frac{\varphi}{2}\right)\right] + \frac{iqp_2}{kp_2'}\varphi - i(s - s_1)\varphi\right\} d\varphi. \tag{9}$$

After substituting Eq. (9) into (3) and integrating over $d^3p_2'ds$, we write Eq. (3) in a form which is convenient for going over to the quasireal-photon approximation:

$$d\sigma = \frac{(\sqrt{4\pi}\,Ze)^2}{4I}\Gamma^{\mu\nu}L_{\mu\nu}\frac{d^4q}{q^4(2\pi)^3 2k(p_2 - q)},$$

$$\Gamma^{\mu\nu} = \frac{1}{2\pi}\int[2mu(-\tau/2) - q]_\mu[2mu(\tau/2) - q]_\nu \times$$
$$\times \exp\left[-\frac{ikp_2}{kp_2'}aq\right]\frac{dkp_2\tau}{m_2}, \tag{10}$$

$$L_{\mu\nu} = \frac{4\pi e^2}{2\lambda + 1}\int(2\pi)^4\delta(p_1 + q + s_1k - p')\,j_\mu(s_1)\,j_\nu^*(s_1)\,d\rho'\frac{ds_1}{\delta(0)},$$

where λ is the spin of particle p_2, and

$$a_\mu = u_\mu\tau + \ddot{u}_\mu\frac{\tau^3}{24} - \frac{q_\mu}{2m_2}\tau.$$

The tensor $L_{\mu\nu}$ can be expanded in invariant structure functions:

$$L_{\mu\nu} = -\Lambda_{\mu\nu}W_1 + \frac{1}{m_1^2}(\Lambda p_1)_\mu(\Lambda p_1)_\nu W_2,$$
$$\Lambda_{\mu\nu} = g_{\mu\nu} - q_\mu q_\nu/q^2 \tag{11}$$

and the latter can be expressed in terms of the cross section of the process $p_1 + q = p'$ with transverse σ_T and longitudinal σ_L virtual photons with the aid of Eqs. (A.1) and (A.2) (see Appendix).

Taking Eq. (A.3) into account, we obtain from Eq. (10)

$$d\sigma = -\frac{(\sqrt{4\pi}\,Ze)^2}{q^4}\frac{I_q}{I}\left\{\left(\Gamma_1 + \frac{q^2\Gamma_2}{I_q^2}\right)\sigma_T + \frac{q^2\Gamma_2}{I_q^2}\sigma_L\right\}\frac{d^4q}{(2\pi)^3 2k(p_2 - q)},$$
$$\Gamma_1 = \Gamma_\mu^\mu,\ \Gamma_2 = \Gamma_{\mu\nu}p_1^\mu p_1^\nu,\ I_q^2 = (p_1q)^2 - m_1^2q^2. \tag{12}$$

When the quasireal-photon approximation is applicable, the spectral distribution over energies is governed by the first term in Eq. (12):

$$dn = -\frac{(\sqrt{4\pi}\,Ze)^2}{q^4}\,\frac{I_q}{I}\left(\Gamma_1 + \frac{q^2\Gamma_2}{I_q^2}\right)\frac{d^4q}{(2\pi)^3\,2k\,(p_2-q)}. \qquad (13)$$

Further, consider in Eq. (13) only the smooth logarithmic contribution. Therefore, in integrating Eq. (13) over the proper time we will drop quantities $\sim \frac{m_2^3 m_1^3 \chi_2}{\eta\nu^2}$ $(\nu = p_1p_2,\ \eta = p_1 q)$.

$$dn = -\frac{2(\sqrt{4\pi}\,Ze)^2}{m_2 q^4}\,\frac{I_q}{I}\left(\gamma_1 + \frac{q^2\gamma_2}{I_q^2}\right)\sqrt{\pi}\,(q\ddot{u}_2)^{-1/3}\times$$
$$\times\left(\frac{\chi_2}{\chi_2'}\right)^{2/3}\frac{d^4q}{(2\pi)^4}\,\Phi(x), \qquad (14)$$
$$x = \left(\frac{\chi_2}{\chi_2'}\right)^{2/3}(q\ddot{u}_2)^{-1/3}\left(qu_2 - \frac{q^2}{2m_2}\right),\ \gamma_{\mu\nu} = (2p_2-q)_\mu(2p_2-q)_\nu.$$

Here $\Phi(x)$ is an Airy function. It is convenient to perform the integration over d^4q in the rest frame of particle p_1 and then to restore the invariant form of notation. In this system

$$p_1 = (m_1, 0),\quad p_2 = (\varepsilon_2, p_2),\ m\ddot{u}_2 = (0, -p_2\omega^2),\ \omega = \chi_2\frac{m_2^2 m_1}{\nu},\ m_2\varepsilon_2 = \nu.$$

The Airy function can be represented in the form

$$\Phi(x) = -\frac{d\Phi_1}{dx},\ \Phi_1 = \int_x^\infty \Phi(x)\,dx,\ \Phi(\infty) = 0,\ \Phi(-\infty) = \sqrt{\pi},$$

where $\Phi_1(x)$ behaves as a step function $\sqrt{\pi}\,\theta(-x-1,02)$ [8]. For this reason one can use a well known method in solid state physics [9] for evaluating the integral (14) over the variable $dq_\parallel$ $(q_\parallel = \mathbf{q}\mathbf{p}_2/p_2)$. Evaluating this integral, we obtain $(s_1 = (p_1+q)^2)$:

$$dn = \frac{(Ze)^2}{2\pi}\left(\frac{\nu}{I}\right)^2\left(1 - \frac{\eta}{\nu} + \frac{I_q^2 m_2^2}{\nu^2 q^2}\right)\frac{ds_1}{I_q}\cdot\frac{dq^2}{q^2}, \qquad (15)$$

where $q_\parallel$ satisfies the equation

$$2p_2q - q^2 = -\left(\frac{q\ddot{u}_2\chi_2'^2}{\chi_2^2}\right)^{1/3} m_2. \qquad (16)$$

The domain of variables s_1 and q^2 can be found from the inequalities

$$(\sqrt{s} - m_1)^2 \geqslant s_1 \geqslant p'^2,\ |q^2_{min}| < |q^2| < |q^2_{max}|,\ (s = (p_1+p_2)^2),$$

where q^2_{min}, and q^2_{max} are determined from the equation

$$0 = q^2\left(1 - \frac{x}{\nu}\right) + \left(\frac{m_2 x}{\nu}\right)^2 + \frac{q^4}{4\nu^2}(2\nu + m_2^2) -$$
$$-2\left[x - \frac{q^2}{2}\left(1 + \frac{m_1^2}{\nu}\right)\right]^{4/3}\left(\frac{\nu\chi_2 m_2^3}{I^3}\right)^{2/3} \quad (2x = s_1 - m_1^2).$$

In the region $|q^2| \ll \eta,\ |q^2_{max}|$, one has $x \sim \eta$ and

$$-q^2_{min} = \frac{m_2^2\eta^2}{\nu}\left[1 + 2\left(\frac{\chi_2\nu}{\eta}\right)^{2/3}\right]\left[\nu - \eta - \frac{4}{3}m_2^2\left(\frac{x\chi_2^2}{\nu}\right)^{1/3}\right]^{-1}. \qquad (17)$$

After integration we obtain the smooth logarithmic contribution

$$dn = \frac{(Ze)^2}{\pi} \ln \left| \frac{q^2_{eff}}{q^2_{min}} \right| \frac{d\eta}{\eta}. \tag{18}$$

The quantity q^2_{ef} is determined by the specific form of the process $p_1 + q = p'$. Relation (18) determines the spectrum of photons equivalent to the field of a fast particle moving in an arbitrary homogeneous external field. We will make several observations:

1. In the derivation of Eq. (18) we took into account the recoil of particle p_2. For this reason, Eq. (18) is applicable in a larger region than the Weizsacker-Williams approximation. 2. The result obtained is valid for arbitrary weakly inhomogeneous electromagnetic field. 3. The frequency η is fixed by the energy of the particles p' produced in the collision. 4. For higher fields, Eq. (18) is still applicable under the condition $\chi_2\nu \gg \eta$. 5. The analogous expression obtained in Ref. [10] in the Weizsacker-Williams approximation is different from Eq. (18), but they agree in the case $\chi_2\nu \gg \eta$.

It is of interest to estimate the spectrum of equivalent photons in the framework of classical field theory.

If it is assumed that the state of particle p_2 does not change during the interaction, it will play the role of a source of a classical field. In this case its behavior is determined by solution of the equation of motion $m\dot{u}^\mu = ZeF^{\mu\nu}u_\nu$, where $F_{\mu\nu}$ is the external field tensor.

The electromagnetic field created by particle p_2 is determined by a vector potential A_2 satisfying d'Alembert's equation:

$$A_\mu(x) = -\frac{4\pi Ze}{2m_2} \int J_\mu(q)\, e^{-iq(x-x_0)} \frac{d^4q}{q^2(2\pi)^4}, \tag{19}$$
$$J_\mu(q) = \int 2m_2 u_\mu(\tau)\, e^{-iq(x(\tau)-x_0)}\, d\tau,$$

where $x = x(\tau)$ is the equation of the trajectory, and $x_{0\mu} = x_\mu(0)$. In the rest frame $p_1 = (m_1, 0)$, one has $x_0 = (0, \rho_x, \rho_y, 0)$. The energy flux created by p_2 is

$$\mathbf{S}(t) = \frac{c}{4\pi}[\mathbf{EH}] = \frac{c}{4\pi}[\mathbf{E}[\beta\mathbf{E}]],\ \beta = \frac{\mathbf{u}_2}{u_2}. \tag{20}$$

From Eqs. (19) and (20) one can find an expression for the energy spectrum of the virtual photons radiated in direction q which differs from Eq. (18) by terms which take into account the recoil of p_2. In the classical theory, one is interested in the flux of virtual photons radiated in the direction of momentum $\mathbf{p}_2$ in the Weizsacker-Williams approximation. The corresponding expression for the number density of the photons can be obtained from the equation

$$\int \hbar q_0 n(q_0)\, dq_0 = \frac{c}{4\pi} \int [\mathbf{E}\mathbf{p}_2]^2 \varepsilon_2 \frac{d\rho_x\, d\rho_y\, dt}{m_2^2 p_2} \qquad (m_1 q_0 = \eta). \tag{21}$$

Substituting the value of E into Eq. (21), we find

$$dn(q_0) = 4(Ze)^2 \frac{\varepsilon_2}{m_2 q_0} \frac{[\mathbf{q}\mathbf{p}_2]^2}{p_2^2} e^{-iqa} \frac{d\tau\, d^4q}{q^4(2\pi)^3}.$$

This expression agrees with Eq. (13) if the recoil of particle p_2 is neglected in the latter.

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Appendix

$$\Pi_T^{\mu\nu} = \sum_{\lambda=1,2} e_\lambda^\mu e_\lambda^\nu = -\Lambda^{\mu\nu} - \frac{q^2}{I_q^2} (\Lambda p)^\mu (\Lambda p)^\nu. \tag{A.1}$$

$$\Pi_L^{\mu\nu} = e_3^\mu e_3^\nu - e_0^\mu e_0^\nu.$$

$$\sigma_T = \frac{1}{4I_q} \frac{1}{2} \Pi_T^{\mu\nu} L_{\mu\nu} = \frac{1}{4I_q} W_1. \tag{A.2}$$

$$\sigma_L = \frac{1}{4I_q} \Pi_L^{\mu\nu} L_{\mu\nu} = \sigma_T + \frac{I_q}{4q^2} W_2.$$

$$L^{\mu\nu} = 4I_q \Pi_T^{\mu\nu} \sigma_T + \frac{4q^2 m^2}{I_q} (\Lambda p)^\mu (\Lambda p)^\nu \sigma_L. \tag{A.3}$$

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