INSTABILITY OF LANGMUIR OSCILLATIONS OF PLASMA IN THE PRESENCE OF A RELATIVISTIC BEAM WITH THERMAL VELOCITY DISPERSION

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We investigate the influence of relativistic effects and radiative damping on beam instability. Expressions are obtained for the increment, the effective excitation width of the waves, and the criteria for the occurrence of instability, enabling one to draw a number of new physical conclusions. In particular, in a dense enough plasma the beam instability can be suppressed by radiative damping. We make numerical estimates. For large values of the plasma density and beam energy the effective excitation width of the waves can be so small that excitation of plasma waves becomes practically impossible.

The instability of plasma-beam systems due to the excitation of high-frequency plasma oscillations was discovered by Akiezer and Fainberg [1] and by Bohm and Gross [2]. This topic has subsequently been explored by a number of authors (see, for example, Ref. [3]). The investigation of the stability of Langmuir oscillations has been done, as a rule, on the basis of the classical Vlasov's equation. In such treatment only the Landau damping has been considered as a natural impediment to the onset of instability in a collisionless plasma. However, as was first shown in Refs. [4] and [5], there is also radiative damping of the waves in a plasma, and in many cases of practical importance this exceeds the Landau damping. It is therefore of interest to treat the instability with allowance for radiative damping.

We will examine the relativistic dispersion equation for longitudinal plasma waves with wave vector **k** parallel to the average velocity **V** of the particles of the beam. In this case one can use the dispersion relation obtained in Ref. [5]:

$$1-\frac{4\pi e^2}{\theta k_1}\int\limits_L \frac{u^0 u_1 F_0}{k_0 u^0+k_1 u^1}\,d\Omega - i\,\frac{8\pi e^4}{3m^2c^4k_1}\int\limits_L \frac{u^0(k_0u_1-k_1u_0)}{k_0u^0+k_1u^1}\,F_0\,d\Omega=0. \tag{1}$$

Here $d\Omega=c^3d^3u/u_0$, u^α is the four-vector velocity of the particles, θ is the temperature of the plasma, $k^\alpha=\{\omega/c,\ k\}$, and F_0 is the relativistic Maxwell distribution function.

Equation (1) can be written in an invariant form valid for any relativistic steady-state distribution:

$$1+4\pi r_0\int\limits_L \frac{u_0u^0+u_1u^1}{(k_0u^0+k_1u^1)^2}\left(1-i\,\frac{2}{3}\,r_0k_0u^\sigma\right)F_0(u)\,d\Omega=0,\quad \left(r_0=\frac{e^2}{mc^2}\right). \tag{2}$$

Relation (2) is written in a coordinate system in which $k^\alpha=\{k^0,\ k^1,\ 0,\ 0\}$.

We will assume, as usual [3], that the equilibrium distribution function of the plasma-beam system is of the form

$$F_0 = f_0^{(1)} + f_0^{(2)}, \tag{3}$$

where

$$f_0^{(1)} = n \exp\left\{-\frac{mc^2}{\theta} u_0\right\} \Big/ 4\pi m^{-1} c\, \theta K_2\left(\frac{mc^2}{\theta}\right), \tag{4}$$

$$f_0^{(2)} = n'_L\left(1-\frac{V^2}{c^2}\right) \exp\left\{-\frac{mc^2}{\theta'_L} u_0 + \frac{mcVu^1}{\theta'_L}\right\} \Big/ 4\pi m^{-1} c \theta'_L K_2 \times$$
$$\times \left(\frac{mc^2}{\theta'_L} \sqrt{1-\frac{V^2}{c^2}}\right). \tag{5}$$

Here $u_0 = \sqrt{1-u_l u^l}$ $(l=1, 2, 3)$; n, θ, n'_L, and θ'_L are the densities of electrons and their temperatures in the plasma and beam, respectively, in a coordinate system S_L in which the plasma is at rest; K_2 is a MacDonald function. Functions (4) and (5) are the relativistic Maxwell distributions of the particles of the plasma and beam in system S_L [6]. Substituting Eqs. (3), (4), and (5) into Eq. (2), we obtain

$$1 + 4\pi r_0 \int\limits_L \frac{u_0 u^0 + u_1 u^1}{(k_0 u^0 + k_1 u^1)}\left(1 - i\frac{2}{3} r_0 k_\sigma u^\sigma\right) f_0^{(1)}\, d\Omega +$$
$$+ 4\pi r_0 \int\limits_L \frac{u'_0 u^{0'} + u'_0 u^{1'}}{(k'_0 u^{0'} + k'_1 u^{1'})^2}\left(1 - \frac{2}{3} r_0 k'_\sigma u^{\sigma'}\right) f_0^{(2)'}\, d\Omega' = 0, \tag{6}$$

where the primed quantities refer to a reference frame S' in which the beam is at rest.

Analysis of Eq. (6) in general form is complicated, because the integrals occurring in the dispersion relation are not expressed in terms of well known special functions. We will confine ourselves to approximate solution of Eq. (6). For this we will expand the integrands in the second and third terms in series in powers of v/c to an accuracy of $(v/c)^3$ and v/c, respectively. The average beam velocity remains an arbitrary relativistic velocity. In addition, we will assume that the phase velocity of the wave $V_p = \omega/k^1$ is much greater than the thermal velocity of the plasma particles $V_T = \sqrt{\theta/m}$, and we will therefore use the expansion [7]:

$$\frac{1}{\omega - k^1 v^1} = \frac{1}{\omega}\left[1 + \left(\frac{k^1 v^1}{\omega}\right) + \left(\frac{k^1 v^1}{\omega}\right)^2 + \left(\frac{k^1 v^1}{\omega}\right)^3 + \dots\right].$$

Using these assumptions, one can reduce Eq. (6) to the form

$$1 - \frac{\Omega^2 - i2\omega(\delta_L + \delta_R)}{\omega^2} - \frac{\omega_p'^2}{\omega'^2} I(a') = 0, \tag{7}$$

where

$$\omega' = \gamma(\omega - kV), \quad \gamma = 1/\sqrt{1-(V/c)^2},$$

$$\Omega^2 = \left(1 - \frac{5}{2}\frac{\theta}{mc^2}\right)\omega_p^2 + \left(3 - \frac{33}{2}\frac{\theta}{mc^2}\right)\frac{\theta}{m}k^2, \quad \left(\omega_p^2 = \frac{4\pi e^2 n}{m}\right), \tag{8}$$

$$\delta_L = \sqrt{\frac{\pi}{8}}\left(\frac{m}{\theta}\right)^{3/2}\frac{\omega_p^4}{k^3}\exp\left\{-\frac{m}{2\theta}\frac{\omega_p^2}{k^2} - \frac{3}{2}\right\} \tag{9}$$

is the Landau damping,

$$\delta_R = \frac{1}{3}\frac{r_0}{c}\omega_p^2\left(1 - 2\frac{k^2\theta}{\omega_p^2 m}\right) \tag{10}$$

is the radiative damping [5], and

$$I(a') = -2a'^2(1 + a'z(a')) - i2\sqrt{\pi}\,a'^3 e^{-a'^2} + i\,\frac{2}{3}\frac{r_0}{c}\,\omega' a' Z(a'),$$

$$a' = \frac{\omega'}{k'}\sqrt{\frac{m}{2\theta'}},\quad k' = \gamma\left(k - \frac{\omega}{c}\frac{V}{c}\right), \tag{11}$$

$$Z(a') = \text{V. p.}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{e^{-x^2}}{x - a'}$$

is a probability integral of imaginary argument [8]. For beams whose density n' is significantly smaller than the plasma density n, the last term in Eq. (7) will be much smaller than the second. In first approximation we have

$$\omega_{(1)}^2 = \Omega^2.$$

The second integral gives

$$\omega_{(2)}^2 = \Omega^2 - 2i\omega_{(1)}(\delta_L + \delta_R) - \omega_{(1)}^2\frac{\omega_p'^2}{\omega_{(1)}'^2}\left[i2\sqrt{\pi}\,a_{(1)}'^3 e^{-a_{(1)}'^2} - i\,\frac{2}{3}\frac{r_0}{c}\,\omega_{(1)}' a_{(1)}' Z(a_{(1)}')\right], \tag{12}$$

where

$$a_{(1)}' = a'|_{\omega=\Omega}.$$

From Eq. (12) for $\delta = \operatorname{Im}\omega_{(2)}$ we obtain the expression

$$\delta = -\delta_L - \delta_R - \Omega\frac{\omega_p^2}{\omega_{(1)}'^2}\left[\sqrt{\pi}\,a_{(1)}'^3 e^{-a_{(1)}'^2} - \frac{1}{3}\frac{r_0}{c}\,\omega_{(1)}' a_{(1)}' Z(a_{(1)}')\right]. \tag{13}$$

This formula is obtained for a resonant beam with $|V_p - V| \leqslant \sqrt{\theta'/m}$, i.e., for $|a'| \leqslant 1$.

Using the tables of values of the function $Z(a')$ in Ref. [8] one can easily arrive at the conclusion that for $|a'| \lesssim 1$ and wavelengths of the plasma waves $\lambda \gg r_0\theta'/mc^2$, the second term in the square brackets in Eq. (13) is much smaller than the first. In addition, for $|a'| \lesssim 1$ we have $1 - V^2\Omega/c^2kV \cong 1/\gamma$.

Then

$$\delta = -\delta_L - \delta_R - \Omega\sqrt{\frac{\pi}{8}}\,\gamma^{3/2}\,\frac{\omega_{pL}'^2}{k^3}\left(\frac{m}{\theta_L'}\right)^{3/2}\times$$

$$\times(\Omega - kV)\exp\left[-\gamma\frac{(\Omega - kV)^2 m}{k^2 2\theta_L'}\right]. \tag{14}$$

Here we have used the law of Planck and Laue for the transformation of the temperature, $\theta' = \theta_L'\gamma$ [6].

The difference between Eq. (14) and the analogous nonrelativistic formula (Ref. [7], p. 366) is due to three independent causes. First of all, in Eq. (14) we have taken into account the relativistic corrections for the thermal velocities of the plasma particles to an accuracy of v^3/c^3. This leads to an increase in the increment δ in comparison with its nonrelativistic value, since $\Omega \neq \omega_p$ (see Eq. (8)). Further, the presence of a relativistic beam in the plasma leads to an increase in the increment by a factor of $\gamma^{3/2} = (E/mc^2)^{3/2}$ in the resonance region. In addition, the increment is smaller than the nonrelativistic one by the radiative damping δ_R.

In order to estimate the interval of beam velocities V or phase velocities of the waves within which it is possible to have increasing wave amplitudes, we introduce the concept of an effective excitation width for longitudinal plasma waves. The lower boundary of this interval will be determined from Eq. (14) by replacing the exponential by unity and setting $\delta = 0$. The upper boundary will be determined from the condition $a' = 1$, i.e., from the condition that $\delta+\delta_L+\delta_R$ be reduced by a factor of e. We obtain as a result

$$\frac{1}{\gamma^{3/2}}\sqrt{\frac{2}{\pi}}\,\frac{\delta_L+\delta_R}{\Omega}\,\frac{n}{n_L'}\left(\frac{V_{TL}'}{V_p}\right)^2 < \frac{|\Omega/k - V|}{V_{TL}'} < \sqrt{\frac{2}{\gamma}}. \qquad (15)$$

According to what we have said above, relativistic effects lead to a decrease in the effective excitation width of longitudinal plasma waves at large thermal velocities of the plasma and relativistic beam velocities in comparison with the width of the analogous curve in the nonrelativistic theory.

As was shown in Ref. [6], for a wide interval of electron densities (10^9–10^{15} cm^{-3}) the radiative damping significantly exceeds the Landau damping, beginning with phase velocities of $V_p \sim 10\, V_T$. Thus, within the limits of applicability of the above calculations one should set $\delta_L = 0$ in Eqs. (14) and (15).

The presence of radiative damping in the plasma leads to a dependence of the lower boundary of the effective excitation width of longitudinal plasma waves on the energy of the particles of the beam, i.e., to a dependence of the form $\sim (mc^2/E)^{3/2}$. Such a dependence can apparently be checked experimentally.

We will find the maximum value of the increment δ. It is easy to see that the maximum destabilizing effect of the beam occurs for

$$\gamma\,\frac{(\Omega - kV)}{k^2}\,\frac{m}{2\theta_L'} = \frac{1}{2}. \qquad (16)$$

Therefore

$$\delta_{max} = -\delta_R + \Omega\sqrt{\frac{\pi}{8e}}\,\gamma\,\frac{n_L'}{n}\,\frac{\omega_p^2 m}{k^2\theta_L'}. \qquad (17)$$

A necessary condition for the occurrence of beam instability is the condition $\delta_{max} > 0$. For $\Omega \cong \omega_p$ this condition can be written in the form (in the cgs system)

$$\gamma\,\frac{n_L'}{n^{3/2}}\left(\frac{V_p}{V_{TL}'}\right)^2 > 4{,}3\cdot 10^{-18}. \qquad (18)$$

For example, for $(V_p/V'_{TL}) = 10^2$, $n_L' = 10^3$ cm^{-3}, and $\gamma \sim 1$ we find that the radiative dissipation of the energy of collective plasma waves suppresses beam instability beginning at a plasma particle density of $n \cong 3 \times 10^{15}$ cm^{-3}.

Thus, we have investigated relativistic and radiative-damping effects on beam instability and obtained expressions for the increment and the effective excitation width of waves, and also the condition for the occurrence of beam instability. For several important plasma-beam systems the formulas obtained lead to new physical results. In particular, in a dense enough plasma the beam instability can be suppressed by radiative damping. At large values of the plasma density and beam energy, the effective excitation width of waves can be so small that excitation of plasma waves becomes practically impossible.

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