ON THE MUTUAL SYNCHRONIZATION OF TWO-SPIN OSCILLATORS

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The problem of the stability of generation of a spin oscillator with an inhomogeneously broadened line of the working material is examined for the example of two spin oscillators coupled by a common transverse magnetic field. The single-frequency regimes are investigated in detail for equal and unequal transverse and longitudinal relaxation times and nonzero phase angle in the feedback circuit. Particular attention is paid to the study of the nature of the beats outside the region of stable single-frequency operation.

In this paper we examine the problem of the stability of generation of a spin oscillator [1] with an inhomogeneously broadened line of the working material. The broadening occurs due to the nonuniformity of the magnetic field H_0. Because the study of the stability of generation in the case of an arbitrary distribution of the field over the sample is a rather complicated mathematical problem, in this paper we will treat only the case in which half of the sample is acted upon by the field $H_0 + \Delta H$, while the other half is acted upon by $H_0 - \Delta H$. Thus, the problem reduces to the study of the regimes of generation of two coupled spin oscillators, each of which is described by a Bloch equation [2], while the coupling between them is due to the common transverse magnetic field. The nature of the stability of single-frequency generation has been investigated several times [3-7] for a spin oscillator in a nonuniform magnetic field with a symmetric or asymmetric distribution function for equal or unequal relaxation times and a zero phase angle in the feedback circuit. In Ref. [8] the criterion determining the boundaries of the stability region of the spin oscillator was obtained for arbitrary phase angle and nonlinear feedback. In the present paper, particular attention is devoted to the investigation of the effect of a phase shift in the feedback circuit on the stable single-frequency generation regime and on the nature of the beats outside the region of stable single-frequency operation.

Seeking a solution of the Bloch equations for the i-th oscillator in the form

$$M_{xi} = a_i(t)\cos[\Omega t + \varphi_i(t)], \quad M_{zi} = M_0 b_i(t), \; i = 1, 2, \dots, \tag{1}$$

we obtain for slowly varying a_i, b_i, and φ_i the following abridged equations:

$$\begin{aligned} \frac{da_1}{dt} &= -a_1 + kb_1[a_1\cos\theta + a_2\cos(\theta+\psi)], \\ \frac{da_2}{dt} &= -a_2 + kb_2[a_2\cos\theta + a_1\cos(\theta-\psi)], \\ \frac{db_1}{dt} &= \delta(1-b_1) - ka_1[a_1\cos\theta + a_2\cos(\theta+\psi)], \end{aligned} \tag{2}$$

$$\frac{db_2}{dt} = \delta(1 - b_2) - ka_2[a_2\cos\theta + a_1\cos(\theta - \psi)],$$

$$\frac{d\psi}{dt} = -\frac{\xi}{2} + k\sin\theta(b_2 - b_1) + + k\left[b_2\frac{a_1}{a_2}\sin(\theta - \psi) - b_1\frac{a_2}{a_1}\sin(\theta + \psi)\right]. \quad (2)$$

Here

$$\xi = \frac{(\omega_2^2 - \omega_1^2)}{\Omega^2\delta_2},\ \tau = \Omega\delta_2 t,\ \delta = \frac{\delta_1}{\delta_2},\ k = \frac{\gamma\alpha}{2M_0\Omega},\ \psi = \varphi_2 - \varphi_1,$$

ω_i is the frequency of autonomous oscillations of the i-th oscillator, $1/\delta_2$ is the transverse relaxation time, $1/\delta_1$ is the longitudinal relaxation time, θ is the phase advance in the feedback circuit, γ is the gyromagnetic ratio, α is the amplification factor, M_0 is the equilibrium magnetization, and Ω is the as yet unknown frequency of the synchronous single-frequency oscillations.

The frequency detuning between the oscillators is governed by the quantity ξ.

To determine the frequency of synchronous oscillations, one can write the following algebraic equation:

$$\frac{\eta}{2} = kb_2\left[\sin\theta + \left(\frac{a_1}{a_2}\right)\sin(\theta - \psi)\right], \quad (3)$$

where $\eta = \frac{(\Omega^2 - \omega_2^2)}{\Omega^2\delta_2}$.

In the general case of arbitrary system parameters δ, ξ, k, and θ, it is not possible to find the steady-state solution of the system of Equations (2), (3) and to investigate its stability in analytical form. We will therefore examine the particular case $\delta = 1$, $\theta \neq 0$, i.e., the transverse and longitudinal relaxation times are identical, but there is a phase advance in the feedback circuit.

In this case the solution of the system of transcendental Equations (2), (3) can be reduced to the solution of the following equation for the unknown reduced frequency η of synchronous oscillations:

$$\left[\left(\eta - \frac{\xi}{2}\right)\cos\theta - 2\sin\theta\right][\eta^2\cos\theta - 2\eta\sin\theta - \eta\xi\cos\theta + 4k + + 2\xi\sin\theta] = \left(\frac{\xi^2}{2}\right)\sin\theta\cos\theta. \quad (4)$$

Equation (4) is a quadratic equation in ξ and has two roots. However, only the root which tends to 2η as $\theta \to 0$ [3] has physical meaning, i.e.,

$$\xi = \frac{3}{2}\eta - \mathrm{tg}\,\theta + \frac{2k}{\eta\cos\theta - 2\sin\theta} + + \sqrt{\left(\frac{\eta}{2} + \mathrm{tg}\,\theta - \frac{2k}{\eta\cos\theta - 2\sin\theta}\right)^2 + \frac{16\,\mathrm{tg}\,\theta k}{\eta\cos\theta - 2\sin\theta}} \quad (5)$$

The boundaries of the mode-locking region can be calculated on a computer. Figure 1 shows the results of such a calculation for θ = 0.5, 10, 20, and 45°. The lower boundary of the synchronous-oscillation regime corresponds to the condition for self-excitation of the oscillations, or $a_1 = a_2 = 0$. In all the graphs there is a widening of the single-frequency mode-locked operating region at two

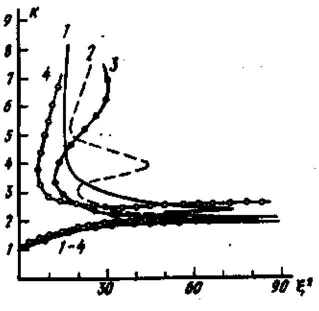

Fig. 1

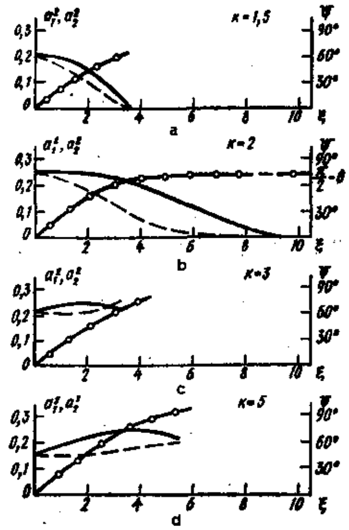

Fig. 2

Fig. 1. Boundaries of the mode-locked region for four values of the phase shift in the feedback circuit: $\theta = 0.5^\circ$ (1), 10° (2), 20° (3), 45° (4).

Fig. 2. Squares of the amplitudes a_1^2 and a_2^2 of the oscillations of the generators and their relative phase shift ψ as functions of the detuning ξ for various values of the amplification factor k at a phase shift in the feedback circuit $\theta = 15^\circ$; solid curve is a_1^2, dashed curve is a_2^2, line with circlets is ψ.

values of the feedback coefficient k. One such widening occurs at $k = 2/\cos\theta$, at which $\xi \to \infty$. This strong broadening of the mode-locked band occurs at the threshold of excitation, where the conditions for generation are satisfied for one of the generators but not for the other. Thus, we have not a regime of synchronous oscillations but induced single-frequency oscillations. This phenomenon is well illustrated in Fig. 2b. An analogous phenomenon of unstable single-frequency generation for large inhomogeneous broadening of the field was obtained by Vladimirskii [3] for the case of three oscillators, also at the excitation threshold. We note that for $k = 2/\cos\theta$, one has $\psi = \pi/2 - \theta$ at $\xi \to \infty$.

The second widening of the single-frequency generation region is observed at large values of k. We have not succeeded in obtaining an analytical expression for the corresponding value of k. However, as calculations on a digital computer have shown, for this value of k the angle ψ is equal to $\pi/2$ at the boundary of the mode-locked region. For small values of θ the second widening of the mode-locked region occurs for k close to $2/\cos\theta$. As θ increases, the onset of this widening is shifted to larger k: for $\theta = 10$, 20, and 45° the widening is maximum for $k = 4$, $k = 6.5$, and $k > 6.5$, respectively.

We will now examine the beats outside the stable single-frequency generation region for the case $\theta = 0$ and $\delta \neq 1$. The conditions for stable single-frequency generation for this particular case have been obtained in Refs. [3-7]. Before proceeding to the investigation of the beats, we will point out that for $\theta = 0$ one has by virtue of the symmetry of the abridged Equations (2)

$$a_1(t) = a_2(t) = a(t), \quad b_1(t) = b_2(t) = b(t). \tag{6}$$

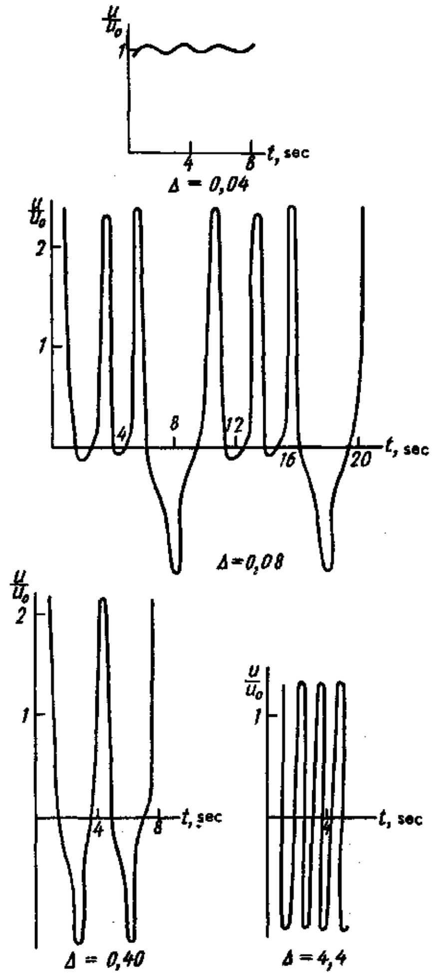

Fig. 3

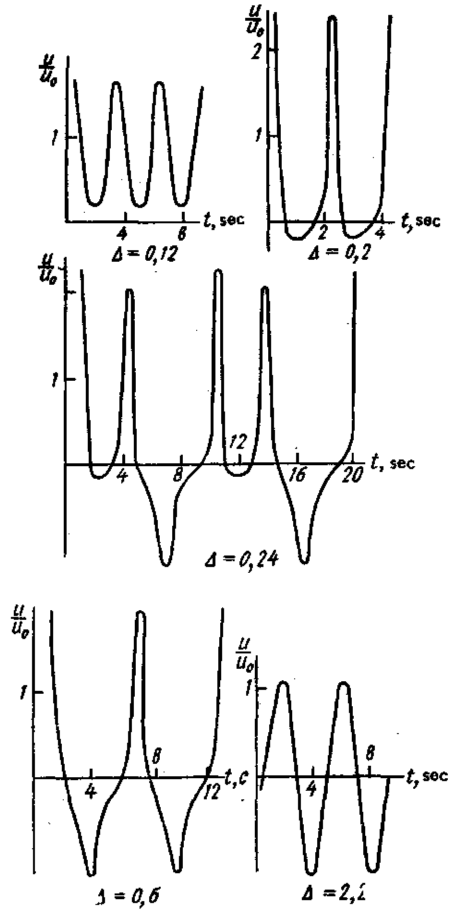

Fig. 4

Fig. 3. Graphs of $u(t)/u_0$ (u_0 is the value of $u(t)$ at the boundary of the mode-locked region) outside the mode-locked band for various values of the detuning Δ between the oscillator for $\delta = 1$, $\xi_b = 4$, $k = 5$, $u_0 \simeq 0.61$.

Fig. 4. Graphs of $u(t)/u_0$ outside the mode-locked band for various values of the detuning Δ between oscillators for $\delta = 0.5$, $\xi_b = 2.8$, $k = 5$, $u_0 \simeq 0.24$.

It is convenient to make the change of variables:

$$u = a\cos\left(\frac{\psi}{2}\right), \quad v = a\sin\left(\frac{\psi}{2}\right).$$

Then Equations (2) can be rewritten in the form

$$\begin{aligned} \dot{u} &= -u + 2kbu + \left(\frac{\xi}{4}\right)v, \\ \dot{v} &= -v - \left(\frac{\xi}{4}\right)u, \\ \dot{b} &= \delta(1-b) - 2ku^2. \end{aligned} \tag{7}$$

In the feedback coil one observes a signal

$$a_1 \cos \Omega t + a_2 \cos (\Omega t + \psi) = 2u(t) \cos \left(\Omega t + \frac{\psi}{2}\right).$$

Consequently, the function 2u(t) is the envelope of the high-frequency field observed in the experiment. The beats correspond to the self-oscillations in the system of Equations (7). The frequency and amplitude of the self-oscillations can be found analytically in two limiting cases, for which they are sinusoidal:

1. $\Delta = \xi - \xi_b \ll \xi_b$ - beats at the boundary of the mode-locked region. Such beats were examined in Refs. [3,7]. Their amplitude grows as $\sqrt{\Delta}$, and the frequency is equal to zero at k = 2 and increases with the growth of k.

2. $\xi > \xi_b$ - the lines are well resolved and are located rather far apart.

This case was examined in Ref. [9]. Under these conditions system (7) has only one unstable stationary state:

$$u = v = 0, \; b = 1. \tag{8}$$

Seeking a solution of system (7) in the form

$$v = A(t) \cos \left(\frac{\xi}{4} + \varphi(t)\right), \quad u = A(t) \beta \cos \left(\left(\frac{\xi}{4}\right) t + \varphi(t) + \theta\right), \quad b = b(t),$$

we obtain [10] the following equations for A(t), b(t), and φ(t):

$$\begin{aligned} \frac{dA}{dt} &= -A[1 - kb], \\ \frac{db}{dt} &= \delta(1 - b) - \frac{16k}{\xi^2} A^2 \left(1 + \frac{\xi^2}{16}\right), \\ \frac{d\varphi}{dt} &= -\frac{4k}{\xi} b, \text{ where } \beta^2 = \left(1 + \frac{\xi^2}{16}\right), \; \mathrm{tg}\,\theta = -\frac{\xi}{4}. \end{aligned} \tag{9}$$

The steady-state beats of the variable u have an amplitude given by the equation

$$A^2 = \frac{\delta(1 - 1/k)}{\beta^2 \cdot 16k} \xi^2, \tag{10}$$

and a frequency of

$$\Omega = \frac{\xi}{4} + \dot{\varphi} \approx \frac{\xi}{4}. \tag{11}$$

For the case of intermediate ξ there are no regular methods of evaluating the amplitude and frequency of the limit cycle of system (7). We have therefore investigated system (7) on an analog computer. The results are shown in Figs. 3 and 4. It can be seen from the figure that the nature of the beats is qualitatively little different in the cases $\delta = 1$ and $\delta \neq 1$. We note that the oscillations of the generators outside the mode-locked band are modulated both in amplitude and in frequency.

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