THE OPERATIONS OF RECIPROCATION AND INVERSION OF COMPLEX PHYSICAL QUANTITIES

V. I. Shestakov

Vestnik Moskovskogo Universiteta. Fizika,
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The operations of reciprocation V^{-1} and inverse V' of indefinite complex physical quantities are defined and investigated. It is noted that the operations of additions, inverse (harmonic) addition $V_1 \bullet V_2 = (V_1' + V_2')'$, and inversion of indefinite complex quantities V_1 and V_2 are generalizations of the basic operations of Boolean algebra. At the end of the article, we examine the question of whether the inversion operation can actually (physically) be carried out for any complex (and in particular, real) physical quantities.

§ 1. NUMERICAL VALUES OF PHYSICAL QUANTITIES

Let V_e be a positive scalar quantity [1], understood to be the unit of quantity V. Quantity V is called complex if

$$V = \gamma V_e, \tag{1}$$

where $\gamma = \alpha + j\beta$, α, and β are finite real numbers $j = \sqrt{-1}$, or else $\gamma = \infty$, where ∞ is an improper number which can usually be associated with the set of complex numbers. The quantity V is called imaginary if $\alpha = 0$, $\beta \neq 0$ and real if $\alpha \neq 0$, $\beta = 0$.

The numbers 0 and ∞ are usually called singular in view of their special role in algebraic operations. The expressions $\infty - \infty$, $0 \cdot \infty$, $\infty \cdot 0$, $\frac{0}{0}$, $\frac{\infty}{\infty}$ are usually considered indeterminate, whereas similar expressions for all the other numbers have unambiguous, definite meanings. The values $0V_e$ and ∞V_e of the quantity V will be called its singular or degenerate values. We will call the variable quantity with the domain of values $\{0V_e, \infty V_e\}$ a degenerate variable and denote it by the lower-case letter v. In particular, we will call a numerical variable with the domain of values $\{0, \infty\}$ a degenerate numerical variable.

If the system of units of the physical quantities is fixed and, hence, the unit V_e of the quantity V is fixed, then it is useful to introduce for denoting the numerical value γ of the quantity V the special symbol „V", defined by the formula

$$„V" \underset{D}{=} V/V_e, \tag{D1}$$

where the sign $\overset{=}{D}$ denotes equality by definition.

Setting in Equation (1) γ = „V", we obtain the equation

$$V = „V" V_e. \qquad (2)$$

If V_e = 1, i.e., if V is a dimensionless quantity (number), then

$$V = „V". \qquad (2_0)$$

For example, Equation (2_0) is valid if V is the amplification factor of the current, voltage, or power of an amplifier or the transverse or longitudinal magnification of an optical system. For nondimensionless physical quantities an equation of the form (2_0) lacks meaning. In particular, even the equations

$$0\, V_e = 0, \quad \infty V_e = \infty \qquad (3)$$

are meaningful only if V_e is a number.

The dimensionality [V] of any physical quantity V does not depend, of course, on its numerical value „V" and therefore remains the same for both nondegenerate and degenerate values of the quantity V.

For equality

$$V = W \qquad (4)$$

of the physical quantities $V = „V" V_e$ and $W = „W" W_e$ it is necessary and sufficient that they satisfy the two equations:

$$„V" = „W", \quad V_e = W_e. \qquad (5)$$

The first of these will be called the numerical equality of the quantities V and W. Equation (4) lacks meaning if the quantities V and W have different dimensions.

If V and W are numerically equal, i.e., satisfy the first of Equations (5), then obviously

$$V = „W" V_e, \quad W = „V" W_e. \qquad (6)$$

It is also clear that, conversely, the first of Equations (5) is implied by equation (6).

The quantity V which is numerically equal to W will be denoted by the symbol $V_{„W"}$, which is defined by the formula

$$V_{„W"} \underset{D}{=} „W" V_e. \qquad (D2)$$

This symbol will denote only those of V which satisfy the first of Equations (6) for some given value of W. The symbol $W_{„V"}$ has an analogous meaning.

One is readily convinced that

$$V_{„W"} \cdot W_{„V"} = W \cdot V. \qquad (7)$$

In fact,

$$V_{„W"} \cdot W_{„V"} = „W" V_e \cdot „V" W_e = „V" W_e \cdot „W" V_e = W \cdot V.$$

§2. RECIPROCATION OF NUMBERS AND PHYSICAL QUANTITIES

The operation γ^{-1} of transforming a number γ to its reciprocal $1/\gamma$ will be called the operation of reciprocation of the number γ. We will extend this operation to the numbers 0 and ∞ by means of the following definition:

$$\gamma^{-1} \underset{D}{=} \begin{cases} 1/\gamma, & \text{if } \gamma \neq 0 \text{ и } \gamma \neq \infty, \\ 0, & \text{if } \gamma = \infty, \\ \infty, & \text{if } \gamma = 0. \end{cases} \tag{D3}$$

The operation V^{-1} defined by the formula

$$V^{-1} \underset{D}{=} ,,V'',^{-1} \cdot V_e^{-1}, \tag{D4}$$

where V_e^{-1} is the unit which is reciprocal to V_e, will be called the operation of reciprocation of the quantity V.

The definition (D4) implies the equation

$$(\gamma V)^{-1} = \gamma^{-1} V^{-1}, \tag{8}$$

which asserts that the operation of reciprocation is distributive with respect to the multiplication γV of any number γ by any finite positive quantity V.

Definition (D4) also implies

$$v^{-1} = ,,v'',^{-1} V_e^{-1}, \tag{9}$$

from which, by virtue of (D3), follow the equations

$$(0V_e)^{-1} = \infty V_e^{-1}, \quad (\infty V_e)^{-1} = 0 V_e^{-1}. \tag{10}$$

Definitions (D3) and (D4) also imply the involutivity of the operation of reciprocation, i.e., the validity of the equations

$$(\gamma^{-1})^{-1} = \gamma, \quad (V^{-1})^{-1} = V \tag{11}$$

for any values of γ and V.

When one of the numerical variables γ and δ assumes the value 0 while the other assumes the value ∞, their product becomes indefinite: $\gamma\delta = 0 \cdot \infty$ or $\gamma\delta = \infty \cdot 0$. This indeterminacy cannot be resolved if the variables γ and δ are independent, for in this case the product $\gamma\delta$ does not have a limit as the variables γ and δ tend to different values 0 and ∞. However, if $\delta = \gamma^{-1}$, the equations

$$\gamma\gamma^{-1} = \gamma^{-1}\gamma = \frac{\gamma}{\gamma} = 1 \tag{12}$$

of course remain valid even in the limiting cases $\gamma \to 0$ and $\gamma \to \infty$. Hence, by virtue of (D4), it follows that the equations

$$VV^{-1} = V^{-1}V = \frac{V}{V} = 1 \tag{13}$$

are valid for any values of V.

§3. INVERSION OF PHYSICAL QUANTITIES

The operation V' defined by the formula

$$V' \underset{D}{=} V^{-1} V_e^2, \tag{D5}$$

will be called the inversion of the quantity V.

Definitions (D5) and (D4) imply the equation

$$V' = {}_{,,}V^{,,-1} V_e, \tag{14a}$$

which is obviously equivalent to the equation

$$V' = {}_{,,}V^{-1,,} V_e. \tag{14b}$$

By virtue of (D2) this equation is equivalent to

$$V' = V_{,,V^{-1},,}. \tag{15}$$

Equations (14a), (14b), and (15) assert (in somewhat different form) that the inverse V' of the quantity V is numerically equal to 1/V, the reciprocal of V. This assertion is valid, in particular, for degenerate values v of the quantity V:

$$v^{-1} = v_{,,v^{-1},,}. \tag{16}$$

In fact, substituting into Eq. (14a) the values of the degenerate variable v, we obtain the equations

$$(0V_e)' = \infty V_e, \quad (\infty V_e)' = 0V_e, \tag{17}$$

the right-hand sides of which are numerically equal to right-hand sides of the respective Equations (10).

The involutivity of the reciprocation of numbers, i.e., the first of Equations (11), implies the involutivity of the inversion operation:

$$(V')' = V. \tag{18}$$

In fact:

$$(V')' = ({}_{,,}V^{,,-1} V_e)' = ({}_{,,}V^{,,-1})^{-1} V_e = {}_{,,}V^{,,} V_e = V.$$

It should be noted that the operations of reciprocation and inversion are permutable:

$$(V')^{-1} = (V^{-1})'. \tag{19}$$

In fact:

$$(V')^{-1} = ({}_{,,}V^{,,-1} V_e)^{-1} = ({}_{,,}V^{,,-1})^{-1} V_e^{-1} = ({}_{,,}V^{,,-1} V_e^{-1})' = (V^{-1})'.$$

As is evident from formula (D5),

$$V' = V^{-1} \tag{20}$$

only in the case when $V_e = 1$ and, consequently,

$$\Upsilon' = \Upsilon^{-1}. \tag{20_0}$$

Using successively Equations (2), (14a), (8), (14a) and (20_0), we obtain the equation

$$(\gamma V)' = \gamma' V', \tag{21}$$

which asserts that the inversion operation is distributive with respect to multiplication of any number γ by any quantity V.

The main difference between the operations V' and V^{-1} is that the equation $V^{-1} = V$ is meaningless if V is not a number, whereas the analogous equation $V' = V$ has meaning for any physical quantity V, even though it is true only for $V = V_e$.

§4. RECIPROCATION AND INVERSION OF COMPLEX PHYSICAL QUANTITIES

The addition of the complex physical quantities

$$V_1 = \gamma_1 V_e, \quad V_2 = \gamma_2 V_e \tag{22}$$

will be defined by the formula

$$\gamma_1 V_e + \gamma_2 V_e \underset{D}{=} (\gamma_1 + \gamma_2) V_e. \tag{D6}$$

Because the unit V_e of the quantities V_1 and V_2 can be an arbitrary finite positive quantity V of the same kind as V_1 and V_2, it follows immediately from definition (D6) that

$$\gamma_1 V + \gamma_2 V = (\gamma_1 + \gamma_2) V \tag{23}$$

for any finite positive value of V and any numbers γ_1 and γ_2.

Definition (D6) also implies the equation

$$V' + V = (\gamma^{-1} + \gamma) V_e, \tag{24}$$

which is valid for any physical quantity V.

The operation $V_1 \bullet V_2$ defined by the formula

$$V_1 \bullet V_2 \underset{D}{=} (V_1' + V_2')', \tag{D7}$$

will be called inverse addition, and the result of this operation will be called the inverse sum of V_1 and V_2.

Successfully using formulas (D7), (14a), (D6), (14a), (11), (8), (D6), (8), and (22), we obtain the chain of equations:

$$\begin{aligned} V_1 \bullet V_2 &= (V_1' + V_2')' = (\gamma_1^{-1} V_e + \gamma_2^{-1} V_e)' = ((\gamma_1^{-1} + \gamma_2^{-1}) V_e)' = \\ &= (\gamma_1^{-1} + \gamma_2^{-1})^{-1} V_e = (\gamma_1^{-1} + \gamma_2^{-1})^{-1} (V_e^{-1})^{-1} = ((\gamma_1^{-1} + \gamma_2^{-1}) V_e^{-1})^{-1} = \\ &= (\gamma_1^{-1} V_e^{-1} + \gamma_2^{-1} V_e^{-1})^{-1} = ((\gamma_1 V_e)^{-1} + (\gamma_2 V_e^{-1}))^{-1} = (V_1^{-1} + V_2^{-1})^{-1}, \end{aligned}$$

which implies the formula

$$V_1 \bullet V_2 = (V_1^{-1} + V_2^{-1})^{-1}. \tag{D7'}$$

This formula and the meaninglessness of the expressions $V^{-1} + V$ and $V + V^{-1}$ in the case when V is not a dimensionless quantity imply that the expressions $V^{-1} \bullet V$ and $V \bullet V^{-1}$ also lack meaning in this case.

The distributive law (23) implies the distributive (with respect to the inverse addition of numbers) law of multiplication:

$$\gamma_1 V \bullet \gamma_2 V = (\gamma_1 \bullet \gamma_2) V, \qquad (23')$$

which is valid for any positive quantity V and any numbers γ_1 and γ_2. Hence, in particular, we obtain the formula

$$\gamma_1 V_e \bullet \gamma_2 V_e = (\gamma_1 \bullet \gamma_2) V_e,$$

which is analogous to definition (D6). This formula implies the equation

$$V' \bullet V = (\gamma^{-1} \bullet \gamma) V_e, \qquad (24')$$

which is analogous to equation (24).

By virtue of the involutivity of the operations of reciprocation and inversion, formulas (D7) and (D7') imply the formulas

$$(V_1 \bullet V_2)^{-1} = V_1^{-1} + V_2^{-1}, \quad (V_1 \bullet V_2)' = V_1' + V_2'. \qquad (25)$$

Replacing the quantities V_1 and V_2 in the left-hand formula of (25) by V_1^{-1} and V_2^{-1}, respectively, and in the right-hand formula of (25) by V_1' and V_2', we obtain by virtue of the involutivity of reciprocation and inversion the equations:

$$V_1 + V_2 = (V_1^{-1} \bullet V_2^{-1})^{-1}, \quad V_1 + V_2 = (V_1' \bullet V_2')'. \qquad (26)$$

Comparison of these equations with formulas (D7) and (D7') shows that the addition $V_1 + V_2$ of the quantities V_1 and V_2 can be expressed in terms of the operation of inverse addition of V_1^{-1} and V_2^{-1} of V_1' and V_2' in exactly the same way as the inverse addition of $V_1 \bullet V_2$ of V_1 and V_2 can be expressed in terms of the addition of V_1^{-1} and V_2^{-1} or V_1' and V_2'. More concisely, the operations $V_1 + V_2$ and $V_1 \bullet V_2$ are isomorphs of each other. For dimensionless quantities V_1 and V_2 the operation of reciprocation coincides with the operation of inversion, as can be seen from formula (20), and for this reason the difference between formulas (D7) and (D7') vanishes. In this case, as in Ref. [2], the operation $V_1 \bullet V_2$ will be called harmonic addition.

By virtue of the mutual reciprocity of the operations + and $\bullet$, the commutativity and associativity of the operation + imply the commutativity and associativity of the operation $\bullet$:

$$V_1 + V_2 = V_2 + V_1, \qquad V_1 \bullet V_2 = V_2 \bullet V_1, \qquad (A_1)$$

$$V_1 + (V_2 + V_3) = (V_1 + V_2) + V_3, \; V_1 \bullet (V_2 \bullet V_3) = (V_1 \bullet V_2) \bullet V_3. \qquad (A_2)$$

One is readily convinced that the degenerate quantities play opposing roles in the operations + and $\bullet$:

$$0V_e + V = V, \; 0V_e \bullet V = 0V_e, \qquad (27)$$

$$\infty V_e + V = \infty V_e, \; \infty V_e \bullet V = V, \qquad (28)$$

i.e., the quantity $0V_e$, which is the neutral element (zero) of the operation +, is the all-absorbing element (universal absorber) of the operation ●, while the quantity ∞V_e, which is the all-absorbing element of the operation +, is the neutral element of the operation ●.

Using formulas (27) and (28), one is easily convinced of the validity of the law of absorption

$$(V_1 \bullet v_2) + v_2 = v_2, \; (V_1 + v_2) \bullet v_2 = v_2 \qquad (A_3)$$

and of the distributive laws

$$v_1 \bullet (V_2 + V_3) = (v_1 \bullet V_2) + (v_1 \bullet V_3), \; v_1 + (V_2 \bullet V_3) = (v_1 + V_2) \bullet (v_1 + V_3), \qquad (A_4)$$

where v_1 and v_2 are degenerate quantities and V_1, V_2, and V_3 are any physical quantities.

Replacing the arbitrary quantity V in Equations (24) and (24') by a degenerate quantity v and taking into account that $\infty + 0 = \infty$ and $\infty \bullet 0 = \infty$, we obtain the equations:

$$v \bullet v' = 0V_e, \; v + v' = \infty V_e, \qquad (29)$$

which, by virtue of formulas (27) and (28), imply the equations

$$(v \bullet v') + V_2 = V_2, \quad (v + v') \bullet V_2 = V_2. \qquad (A_5)$$

If all the quantities V_1, V_2, and V_3 are replaced by degenerate quantities v_1, v_2, and v_3, respectively, then formulas (A_1)-(A_5) given above are the axioms of a Boolean algebra [3] whose basic operations $A \cup B$, $A \cap B$, and -A correspond to operations $v_1 + v_2$, $v_1 \bullet v_2$, and v_1' on the degenerate physical quantities v_1 and v_2.

In other words, the operations of addition $V_1 + V_2$, inverse addition $V_1 \bullet V_2$, and inversion V' for the degenerate values v_1, v_2, and v of the physical quantities V_1, V_2, and V "degenerate" into the operations of Boolean addition $\cup$, Boolean multiplication $\cap$, and the cofactor operations - ( ) of a two-valued Boolean algebra, respectively; that is, the algebra of degenerate physical quantities is a two-valued Boolean algebra.

An algebra whose system of axioms is the set of formulas (A_1)-(A_5) can be called a partially Boolean algebra. Just as in Boolean algebra, the number of basic operations in a partially Boolean algebra can be reduced to two; its basic operations can be considered either addition and inversion or inverse addition and inversion, for the operations + and ● can be expressed in terms of each other and the inversion operation. It should be noted that the operation of inversion cannot be replaced by reciprocation, as the expressions $V + V^{-1}$ and $V \bullet V^{-1}$, as was mentioned previously, do not have meaning for any quantity V which is not a number.

§5. ON THE ACTUAL PERFORMANCE OF THE INVERSION OPERATION ON PHYSICAL QUANTITIES

The operation of reciprocation V^{-1} of some parameter V transforms it to its reciprocal parameter 1/V; this, of course, in no way alters the physical system characterized by the parameter V. Performing the V^{-1} operation changes only the mathematical description of the physical system. For example, by replacing the resistance R, capacitance C, and inductance L of some oscillating circuit by their reciprocal parameters 1/R, 1/C, and 1/L we do not change the oscillating circuit; only the equations describing this current and its oscillations are changed.

The operation of inversion V' of parameter V consists of the replacement of its value $„V"V_e$ by a new value $V_{„V^{-1}"}$, as is described symbolically by equation (15). If, for example, we take the admittance Y of some variable-current network as the paramater V, then Equation (15) will assume the form

$$Y' = Y_{„Z"}, \qquad (30)$$

where $Z = Y^{-1}$ is the impedance of the same network. The inversion operation Y' in this case is the replacement of a network with admittance Y by a new network with admittance $Y_{„Z"}$ which is numerically equal to the impedance Z of the former network. The operation of inversion $(Y_{„Z"})'$ on the network just obtained transforms it back to the original network:

$$(Y_{„Z"})' = Y. \qquad (31)$$

Figure 1 shows a scheme by means of which the operation of inversion of the admittance can be realized. This operation is realized in this case by the switching of switch S; switching it from the left position, indicated in Fig. 1,a by the

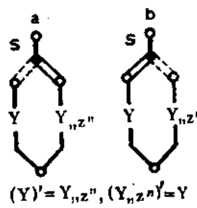

Fig. 1

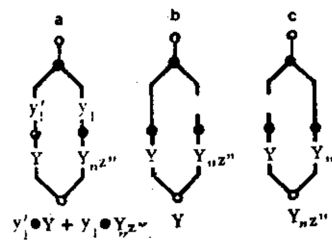

Fig. 2

dashed line, to its right position, we bring about the transformation described by Equation (30). Switching it back causes the reciprocal transformation descrbied by Equation (31), as is shown in Fig. 1,b.

The inversion operation Y' can be automated by using the circuit of Fig. 2,a, in which the switch S of Fig. 1 is replaced by the contacts y_1' and y_1 of some relay R_1. The circuit shown in Fig. 2,a can be represented algebraically by the expression

$$y_1' \bullet Y + y_1 \bullet Y_{„Z"}, \qquad (32)$$

where y_1' and y_1 are the conductivities of the open and closed contacts, respectively, of the relay R_1. The black dots here are simultaneously the signs of inverse addition and the signs of sequential connections $y_1' \bullet Y$ and $y_1 \bullet Y_{„Z"}$ of contacts y_1' and y_1 with the two-terminal networks Y and $Y_{„Z"}$. The + sign in expression (32) serves as the sign of addition of the expressions $y_1' \bullet Y$ and $y_1 \bullet Y_{„Z"}$ and simultaneously denotes that networks $y_1' \bullet Y$ and $y_1 \bullet Y_{„Z"}$ are connected with each other in parallel (Fig. 2,a). In the unexcited state of relay R_1 its opening contact y_1' is closed and its closing contact y_1 is open, i.e., $y_1' = \infty\ \Omega^{-1}$, $y_1 = 0\ \Omega^{-1}$, and consequently,

$$y_1' \bullet Y + y_1 \bullet Y_{„Z"} = \infty_{\text{ohm}^{-1}} \bullet Y + 0\,\text{Om}^{-1} \bullet Y_{„Z"} = Y,$$

i.e., the circuit of Fig. 2,a takes on the form shown in Fig. 2,b. In the excited state of relay R_1, contact y_1' is open and contact y_1 is closed, i.e., $y_1' = 0\ \Omega^{-1}$, $y_1 = \infty\ \Omega^{-1}$, and consequently,

$$y_1' \bullet Y + y_1 \bullet Y_{„Z"} = 0_{\text{ohm}^{-1}} \bullet Y + \infty_{\text{ohm}^{-1}} \bullet Y_{„Z"} = Y_{„Z"},$$

i.e., the circuit of Fig. 2,a takes on the form shown in Fig. 2,c.

Thus, in the transition of relay R_1 from its unexcited state to the excited (operating) state, the inversion operation described by Equation (30) is realized; in the reverse transition, the operation described by Equation (31).

The circuits in Figs. 1 and 2,a only mechanize and automate the process of performing the operation of inversion Y' of admittance Y. The operation can, of course, be performed by replacing the manual two-terminal network with admittance Y by a two-terminal network with a new "inverse" value $Y_{„Z"}$ of the admittance.

An analogous situation occurs in the case of the inversion V' of any complex physical quantity V.

It should be noted that the operation of inversion defined by (D5) is a generalization of the familiar geometric transformation of the inversion of a plane with respect to a circle. In fact, for $V = l$, where l is the length of a rod, filament, wire, or some other physical object, we obtain from formula (D5) the equation

$$l \cdot l' = l_e^2, \qquad \text{(a)}$$

which is analogous to the question

$$OP \cdot OP' = r^2, \qquad \text{(b)}$$

where r is the radius of the circle from the center O, and the point P' is the image of point P, the point P' lying on the straight line OP on the same side of O as P. If we set $l_e = r$, $l = OP$, then $l' = OP'$, i.e., equations (a) and (b) coincide.

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Department of General Physics
for the Faculty of Physics