BRIEF COMMUNICATIONS

EQUATIONS OF MOTION FOR THE DENSITIES OF CLASSICAL DYNAMICAL VARIABLES IN EXTERNAL FIELDS

V. A. Krasnikov

Vestnik Moskovskogo Universiteta. Fizika,
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We obtain in compact form the equations of motion for the densities of the classical dynamical variables in a wide class of external fields. We also examine the linear approximation in the gradients of the external fields.

In a number of problems in nonequilibrium statistical mechanics it is useful to have the equations of motion of the densities of the various dynamical quantities. In particular, for constructing the hydrodynamic equations it is necessary to known the dynamical equations for the quasilocal densities of the integrals of motion - the energy density $\varepsilon(q; x)$, the momentum density $\pi(q; x)$, and the mass density $\rho(q; x)$. For a classical system of N particles interacting with a pair potential $\Phi(|q_i - q_j|)$, these have the explicit form:

$$\varepsilon(q; x) \equiv \gamma_0(q) = \sum_{i=1}^{N} \frac{p_i^2}{2m} \delta(q_i - q) +$$
$$+ \frac{1}{2} \sum_{i<j} \Phi(|q_i - q_j|) \; (\delta(q_i - q) - \delta(q_j - q)), \tag{1}$$
$$\pi_l(q; x) \equiv \gamma_l(q) = \sum_{i=1}^{N} p_i^l \delta(q_i - q), \quad l = 1, 2, 3;$$
$$\rho(q; x) \equiv \gamma_4(q) = m \sum_{i=1}^{N} \delta(q_i - q),$$

where $x = (p_i, q_i)$ is the set of momenta and coordinates of the particles of the system (the particles have mass m).

In Ref. [1] a compact form of the equations of motion for quasilocal operators γ_m ($m=0, 1, 2, \ldots$) was obtained in the quantum-mechanical case. The generalization of these results to the case when external fields are present is given in Refs. [2, 3]. In the present paper this problem is treated by a purely classical approach.

We will examine a classical nonrelativistic statistical system whose Hamiltonian is of the form

$$\mathscr{H}(t) = \int d^3q h(q, t; x),$$
$$h(q, t; x) = \alpha_n \gamma_n(q; x), \quad \alpha_n = \alpha_n(q, t), \quad n = 0, 1, 2, \ldots \tag{2}$$

Here and throughout this paper we use the convenience of summation over repeated indices. We have introduced given external fields $\alpha_n(q, t)$ into the Hamiltonian (2). For the case of no external fields one has in (2).

$$\alpha_0 = 1, \; \alpha_n = 0 \text{ for } n \neq 0.$$

To the dynamical-variable densities $\gamma_m(q; x)$ there correspond (in the sense of representation in the form of a spatial integral of the quasilocal quantities) dynamical variables Γ_m:

$$\Gamma_m = \int d^3q \, \gamma_m(q; x). \tag{3}$$

We will assume that their Poisson brackets are equal to zero:

$$\{\Gamma_n, \Gamma_m\} = 0. \tag{4}$$

Thus, we are restricting ourselves to the case when the densities $\gamma_m(q; x)$ correspond to integrals of motion Γ_m with respect to the free Hamiltonian Γ_0.

Equations (3) and (4) imply that

$$\{\Gamma_n, \gamma_m(q; x)\} = \frac{\partial}{\partial q_k} \gamma_{mnk}(q; x), \quad k = 1, 2, 3, \tag{5}$$

where we have introduced the "current" densities $\gamma_{mnk}(q; x)$ corresponding to the densities of the dynamical quantities $\gamma_m(q; x)$.

The identity

$$\{\Gamma_n, \gamma_m(q)\} + \{\Gamma_m, \gamma_n(q)\} = \frac{\partial}{\partial q_k} \int d^3 q' \, q'_k \times \int_0^1 dg \, \{\gamma_n(q + gq'), \gamma_m(q - (1 - g) q')\} \tag{6}$$

which is valid for any quasilocal dynamical variables γ_m and γ_n will play an important role in what follows.

This identity was proved for the quantum-mechanical case in Ref. [1] (see also Ref. [4]). Since this proof carries over completely to the classical case, we will not reproduce it here. Equations (5) and (6) imply for the "current" densities

$$\gamma_{mnk}(q) + \gamma_{nmk}(q) = \int d^3 q' q'_k \int_0^1 dg \, \{\gamma_n(q + gq'), \gamma_m(q - (1 - g) q')\}, \tag{7}$$

and hence

$$\gamma_{nnk}(q) = \frac{1}{2} \int d^3 q' q'_k \int_0^1 dg \, \{\gamma_n(q + gq'), \gamma_n(q - (1 - g) q')\}. \tag{8}$$

We note that if Γ_m = const, then

$$\gamma_{mnk}(q) = \int d^3 q' \, q'_k \int_0^1 dg \, \{\gamma_n(q + gq'), \gamma_m(q - (1 - g) q')\}. \tag{9}$$

If $\Gamma_l = p_l$ (l = 1, 2, 3), where p_l are the components of the total momentum of the system when the fields are turned off, then

$$\gamma_{lnk}(q) = -\gamma_n(q)\,\delta_{lk} + \int d^3q' q'_k \int_0^1 dg \{\gamma_n(q+gq'),\ \pi_l(q-(1-g)q')\},\quad n = 0, 1, 2, \ldots \tag{10}$$

We will examine the equations of motion for the dynamical-variable densities $\gamma_m(q)$ with Hamiltonian (2)

$$\dot{\gamma}_m(q) = \{\gamma_m(q),\ \mathcal{H}(t)\}.$$

After straightforward manipulations using Eqs. (2) and (7) we obtain

$$\dot{\gamma}_m(q) = -\frac{\partial \alpha_n}{\partial q_k}\gamma_{nmk}(q) - \frac{\partial}{\partial q_k}\alpha_n\gamma_{mnk}(q) - \frac{\partial}{\partial q_k}\int d^3q' q'_k \int_0^1 dg\,(\alpha_n(q+gq',t) - \alpha_n(q,t))\{\gamma_n(q+gq'),\ \gamma_m(q-(1-g)q')\}. \tag{11}$$

We stress that the system of Equations (11) is exact. The form of Equations (11) is especially convenient for application to perturbation theory in the spatial gradients. In particular, when the external fields α_n are weakly inhomogeneous and the expansion

$$\alpha_n(q+gq',t) = \alpha_n(q,t) + g\frac{\partial \alpha_n}{\partial q_l}q'_l + \ldots$$

is valid, it is easily seen that the last term in Eq. (11) is of a higher order in the spatial gradients. Then in the approximation linear in the gradients, the system of Equations (11) assumes the particularly simple form

$$\dot{\gamma}_m(q) = -\frac{\partial \alpha_n}{\partial q_k}\gamma_{nmk}(q) - \frac{\partial}{\partial q_k}\alpha_n\gamma_{mnk}(q). \tag{12}$$

In that particular case when $\gamma_0 = \varepsilon(q)$, $\gamma_l = \pi_l(q)$, and $\gamma_4 = \rho(q)$, we obtain for the nonzero current densities $\gamma_{mnk}(q)$ in Eq. (12), when Eqs. (6), (9), and (10) are taken into account:

$$\begin{gathered}
\gamma_{00k}(q) = \frac{1}{2}\int d^3q' q'_k \int_0^1 dg \{\varepsilon(q+gq'),\ \varepsilon(q-(1-g)q')\},\\
\gamma_{l0k}(q) = -\varepsilon(q)\,\delta_{lk} + \int d^3q' q'_k \int_0^1 dg \{\varepsilon(q+gq'),\ \pi_l(q-(1-g)q')\},\\
\gamma_{40k}(q) = \int d^3q' q'_k \int_0^1 dg \{\varepsilon(q+gq'),\ \rho(q-(1-g)q')\},\\
\gamma_{mlk}(q) = \gamma_m(q)\,\delta_{lk},\\
m = 0, 1, 2, 3, 4;\ l, k = 1, 2, 3.
\end{gathered} \tag{13}$$

Equations (12) together with relations (13) are convenient, in particular, for constructing the hydrodynamic equations of systems in external fields. In the absence of external fields the given choice of γ_m leads to the usual equations of a hydrodynamically ideal liquid.

REFERENCES

1. S. V. Peletminskii and A. I. Sokolovskii, Teor. i. Matem. Fiz., vol. 18, p. 121, 1974.

2. V. A. Krasnikov, Teor. i Matem. Fiz., vol. 39, 1979.
3. V. A. Krasnikov, DAN SSSR, vol. 242, p. 313, 1978.
4. A. I. Akhiezer and S. V. Peletminskii, Methods of Statistical Physics [in Russian], Moscow, 1977.

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