REMARK ON FIELD THEORY IN QUANTIZED SPACE-TIME

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We examine the theory of quantized space-time developed by V. G. Kadyshevskii and co-workers. In this theory momentum space is a surface of constant curvature:

$$(\pi^0)^2-(\pi^1)^2-(\pi^2)^2-(\pi^3)^2+(\pi^4)^2=1.$$

Each value of the physical four-momentum $p^\mu=\pi^\mu$ ($\mu=0, 1, 2, 3$) corresponds to two points in momentum space π and π^*, which differ by the sign of π^4.

Using the conditions of translational invariance and the microcausality of the S matrix, we show that the retarded functions for $n\geq 2$ satisfy the relation

$$R_n(\pi_0, \pi_1, \ldots, \pi_n)=-R_n(\pi_0^*, \pi_1, \ldots, \pi_n),$$

which connects the values of the matrix elements in the region $\pi^4>0$ with the values of the matrix elements in the region $\pi^4<0$.

1. In Refs. [1-14] a scheme for constructing a field theory in quantized space-time was developed. The momentum space R^4 (Minkowski space) was replaced by a surface of constant curvature [1-2]

$$(\pi^0)^2-(\pi^1)^2-(\pi^2)^2-(\pi^3)^2+(\pi^4)^2=1.$$

The Lorentz transformation acts in the usual way on the coordinates π^μ ($\mu=0, \ldots, 3$), leaving π^4 unchanged. For the displacement operator in x space it was proposed to examine the operation of multiplication by $\exp\left(i\sum_{\mu=0}^{3}\pi^\mu a_\mu\right)$.

It is easily seen that two points in a curved momentum space differing only by the sign of π^4 will correspond to identical values of the physical four-momentum $p^\mu(\pi)=\pi^\mu$ ($\mu=0, 1, 2, 3$). In this regard the question arises of how to interpret states with $\pi^4<0$ and how they are connected with the states with $\pi^4>0$. We will show that for $n\geq 2$ there is a symmetry

$$R_n(\pi_0, \pi_1, \ldots, \pi_n)=-R_n(\pi_0^*, \pi_1, \ldots, \pi_n). \quad (1)$$

Here

$$R_n(\pi_0, \pi_1, \ldots, \pi_n) = \left\langle 0 \left| \frac{\delta^n}{\delta\tilde{\varphi}(\pi_1)\ldots\delta\tilde{\varphi}(\pi_n)} \left(\frac{\delta S}{\delta\tilde{\varphi}(\pi_0)} S^+ \right) \right| 0 \right\rangle$$

is the radiation operator, and π^* denotes a point which differs from π by only the sign of π^4. The proof will be based on the condition of microcausality [1]

$$\frac{\delta}{\delta\varphi(\xi)} \left[\frac{\delta S}{\delta\varphi(0)} S^+ \right] = 0 \quad \text{for} \quad \xi \leqslant 0 \tag{2}$$

and translational invariance [1-4]

$$R(\pi_0, \pi_1, \ldots, \pi_n) = \exp\left(i \sum_{i=0}^{n} p(\pi_i)\, a\right) R(\pi_0, \pi_1, \ldots, \pi_n). \tag{3}$$

We note that any condition of microcausality that is stronger than Eq. (2) will also lead to Eq. (1). In particular, relation (1) holds for the version of microcausality proposed in Ref. [3].

2. We will proceeed to prove Eq. (1). Equation (2) implies

$$\int d^4\pi_1 \langle \xi | \pi_1 \rangle \int d^4\pi_0 R(\pi_0, \pi_1, \ldots, \pi_n) = 0 \quad \text{for} \quad \xi \leqslant 0. \tag{4}$$

It is convenient to parameterize the points of the curved momentum space by the coordinates $\mathbf{p} \in R^3$ and $\omega \in [\pi, -\pi]$:

$$\pi_0 = \sqrt{1+\mathbf{p}^2}\sin\omega, \quad \pi^4 = \sqrt{1+\mathbf{p}^2}\cos\omega, \quad \pi^\alpha = \mathrm{p}^\alpha \ (\alpha = 1, 2, 3).$$

In these coordinate the invariant volume element is

$$d^4\pi = \frac{d^4 p(\pi)}{|\pi^4|} = d\omega\, d^3\mathbf{p},$$

and the time operator can be represented as (see Refs. [1, 2]) $\xi_0 = -i\frac{\partial}{\partial\omega}$. Because the region $\xi_0 < 0$ is part of the region $\xi \underset{\sim}{<} 0$, equation (4) entails the equation

$$\int_{-\pi}^{\pi} e^{in\omega} d\omega \int d^4\pi_0 R(\pi_0, \pi(\omega, \mathbf{p}), \pi_2, \ldots, \pi_n) = 0 \quad \text{for} \quad n > 0. \tag{5}$$

As a consequence of translational invariance (3):

$$\begin{aligned} R(\pi_0, \pi_1, \ldots, \pi_n) &= \delta^4(\Sigma p(\pi_i))\, |\pi_0^4| \{\theta(\pi_0^4) R'(\pi_1, \ldots, \pi_n) + \\ &\quad + \theta(-\pi_0^4) R''(\pi_1, \ldots, \pi_n)\}, \\ \int d^4\pi_0 R(\pi_0, \ldots, \pi_n) &= \theta\left(1 - \left(\sum_{i=1}^{n} p(\pi_i)\right)^2\right) \{R'(\pi_1, \ldots, \pi_n) + \\ &\quad + R''(\pi_1, \ldots, \pi_n)\} = G(\pi_1, \ldots, \pi_n). \end{aligned} \tag{6}$$

Now relation (5) can be rewritten in the form

$$\int e^{in\omega} G(\pi_1(\omega, \mathbf{p}), \pi_2, \ldots, \pi_n) = 0 \quad \text{for} \quad n > 0. \tag{7}$$

3. Using Eqs. (6) and (7), we will show that G = 0. For this we will need the following lemma.

Lemma. Suppose that for all positive integers n

$$\int_{-\pi}^{\pi} g(\omega)\, e^{in\omega} d\omega = 0,$$

where g is a generalized function ($g \in D'$). Then, if $g(\omega)$ vanishes on some interval, $g = 0$ everywhere.

Proof. We will examine the function

$$g_\psi(\omega) = \int_{-\pi}^{\pi} g(\omega - \omega')\,\psi(\omega')\,d\omega', \text{ where } \psi \in D.$$

Because of its smoothness, g_ψ can be represented by the absolutely convergent Fourier series [5]:

$$g_\psi(\omega) = \sum_{n=-\infty}^{+\infty} c_n e^{in\omega}, \quad |c_n| \leqslant \frac{\text{const}}{|n|^2}.$$

From the condition of the lemma it is seen that $c_n = 0$ for $n < 0$. This implies that g_ψ is the boundary value of the function $\sum_{n=0}^{\infty} c_n z^n$, which is holomorphic in the disk $|z| < 1$ and continuous on the boundary of this disk. We note that if the carrier ψ is concentrated on a small enough segment, then g will be equal to zero on some interval. Isomorphically transforming the disk $|z| < 1$ into a half-plane and applying Schwarz's lemma and the uniqueness theorem, we arrive at the conclusion that $g_\psi = 0$ if the carrier ψ is concentrated on a segment of sufficiently small size. It follows from this that $g = 0$.

Applying this lemma to formula (7), one can make the following assertion: If $G(\pi_1, \ldots, \pi_n)$ vanishes in the neighborhood of the point $(\pi_1, \ldots, \pi_n)$ and $p(\pi_1) - p(\pi_1')$, then G also vanishes in the neighborhood of the point $(\pi_1', \ldots, \ldots, \pi_n')$. Using the Lorentz invariance, one can generalize this assertion to the case where $p(\pi_1) - p(\pi_1')$ is a time-like vector. But any two points can be joined by a broken line consisting of time-like segments. On the other hand, as a consequence of Eq. (6), G contains the expression $\theta(1-(\Sigma p(\pi_i))^2$ as a multiplicative factor.

Thus, we arrive at the conclusion that $G = 0$. It remains to be noted that by virtue of Eq. (6) the relation $G = 0$ entails the equation

$$R''_{\S}(\pi_1, \pi_2, \ldots, \pi_n) = -R'(\pi_1, \pi_2, \ldots, \pi_n),$$

which also gives symmetry (1).

4. In conclusion we note that for the surface of constant curvature

$$(\pi^0)^2 - (\pi^1)^2 - (\pi^2)^2 - (\pi^3)^2 - (\pi_4)^2 = -1,$$

examined in Ref. [4], one can apply all the arguments used in this paper. Here one will also arrive at Equation (1).

Equation (1) implies that in the theory examined here there are no additional dgrees of freedom due to the region $\pi^4 < 0$.

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