THE PROPERTIES OF SUPERCOHERENT STATES

S. P. Vyatchanin

Vestnik Moskovskogo Universiteta. Fizika,
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Sufficient conditions for a quantum oscillator to be located in a state with a phase uncertainty $\Delta\varphi \approx 1/n$ (n is the average number of photons) are formulated, and a method of measuring the phase of such states is proposed.

In Ref. [1] Boloshin and Gertsenshtein formulated the conditions on the coefficients C_n of the expansion of the wave function in energy eigenstates $|\psi\rangle = \sum_n C_n |n\rangle$, under which the uncertainty of the phase is $\Delta\varphi \approx 1/2n$ and the uncertainty in the number of photons is $\Delta n \approx n$, $n = \langle n \rangle$ (for coherent states $\Delta\varphi \simeq 1/2\sqrt{n}$; $\Delta n \simeq \sqrt{n}$). Such states are called "supercoherent". In this paper we will formulate in other terms the conditions under which $\Delta\varphi \simeq 1/n$ indicate their physical significance, and propose a method of measuring the phase of a supercoherent state.

1. For the sake of definiteness we will examine a mechanical harmonic oscillator with frequency ω. Since the correct introduction of the phase operator is a complicated problem, we will use the operators $\hat{C}$ and $\hat{S}$ [2,3] (the quantum-mechanical analogs of the cosine and sine). It will be assumed everywhere that $n \equiv \langle \hat{n} + 1/2 \rangle \gg 1$ ($\hat{n}$ is an operator of the photon type). Then the operators $\hat{C}$ and $\hat{S}$ can be written in the form of an expansion in powers of $(\hat{n}+1/2)^{-1}$ (we will use the notation $\hat{k} \equiv \hat{n} + 1/2$):

$$\hat{C} = \frac{1}{2}\left\{\frac{1}{\sqrt{\hat{k}}};\ \hat{x}\right\} + \frac{i}{8}\left[\frac{1}{\hat{k}\sqrt{\hat{k}}};\ \hat{y}\right] + \dots \quad (1)$$

$$\hat{S} = \frac{1}{2}\left\{\frac{1}{\sqrt{\hat{k}}};\ \hat{y}\right\} - \frac{i}{8}\left[\frac{1}{\hat{k}\sqrt{\hat{k}}};\ \hat{x}\right] + \dots \quad (2)$$

where $\hat{x} = \frac{1}{2}(a^+ + a)$, $\hat{y} = \frac{1}{2i}(a^+ - a)$, a^+ and a are the creation and annihilation operators, and the brackets {...} and [...] denote the commutator and anticommutator.

It is not hard to show that the rms deviations $\langle \Delta^2\hat{C} \rangle \equiv \langle \hat{C}^2 \rangle - \langle \hat{C} \rangle^2$ and $\langle \Delta^2\hat{S} \rangle$ vary with time according to the simple law [3]:

$$\langle \Delta^2\hat{C} \rangle = \langle \Delta^2\hat{C} \rangle_0 \cos^2\omega t + \langle \Delta^2\hat{S} \rangle_0 \sin^2\omega t + \frac{1}{2}\langle\{\Delta\hat{C};\ \Delta\hat{S}\}\rangle_0 \sin 2\omega t,$$

$$\langle \Delta^2\hat{S} \rangle = \langle \Delta^2\hat{S} \rangle_0 \cos^2\omega t + \langle \Delta^2\hat{C} \rangle_0 \sin^2\omega t - \frac{1}{2}\langle\{\Delta\hat{S};\ \Delta\hat{C}\}\rangle_0 \sin 2\omega t, \quad (3)$$

where $\langle\dots\rangle_0$ denotes an average at the initial instant of time, and $\Delta\hat{C}, \equiv \hat{C} - \langle\hat{C}\rangle$.

From this it is seen that the problem of finding the conditions for supercoherence breaks up into two problems:

a. To find the conditions under which only one phase operator, say $\hat{C}$, obeys $\langle \Delta^2\hat{C}\rangle_0 < 1/n^2$. If $\langle \Delta^2\hat{S}\rangle_0 > 1/n^2$ simultaneously, then at subsequent instants of time $\langle \Delta^2\hat{C}\rangle > \frac{1}{n^2}$.

b. To find the conditions under which $\langle \Delta^2\hat{C}\rangle_0 \simeq \langle \Delta^2\hat{S}\rangle_0 < 1/n^2$, then at subsequent instants of time $\langle \Delta^2\hat{C}\rangle \simeq \langle \Delta^2\hat{S}\rangle < \frac{1}{n^2}$ also.

Of course, the second problem includes the first. We will therefore examine first under what conditons

$$\langle \Delta^2\hat{C}\rangle_0 \leqslant 1/n^2, \tag{4}$$

assuming nothing about the dispersion of operator $\hat{S}$. For a supercoherent state $\langle \Delta^2\hat{n}\rangle_0 \simeq n$. Then it is natural to assume that

$$\left\langle \frac{1}{\hat{k}}\right\rangle \simeq \left\langle \Delta^2\left(\frac{1}{\sqrt{\hat{k}}}\right)\right\rangle \simeq \frac{1}{n}. \tag{5}$$

Writing $\hat{x} = \langle\hat{x}\rangle + \Delta\hat{x}$, and $1/\sqrt{\hat{k}} = \langle 1/\sqrt{\hat{k}}\rangle + \Delta(1/\sqrt{\hat{k}})$ and substituting these into Eqs. (1) with allowance for Eq. (5) and the obvious inequality

$$\langle\{\hat{u};\hat{v}\}\rangle \leqslant 2\sqrt{\langle\hat{u}^2\rangle\langle\hat{v}^2\rangle}, \tag{6}$$

one can show that condition (4) is satisfied for

$$\langle\hat{x}\rangle_0^2 \leqslant 1/8n \text{ and } \langle \Delta^2\hat{x}\rangle_0 \leqslant 1/8n. \tag{7}$$

We will next ascertain under what additional conditions $\langle \Delta^2\hat{S}\rangle_0 < \frac{1}{n^2}$.

It can be shown that when conditions (7) are satisfied

$$\hat{S} = \hat{y}/\sqrt{\hat{y}^2} + O(\hat{k}^{-5/2}).$$

Then the additional condition under which $\langle \Delta^2/S\rangle_0 < 1/n^2$ can be written as

$$\langle\hat{y}/\sqrt{\hat{y}^2}\rangle_0^2 \equiv \langle \mathrm{sign}(\hat{y})\rangle_0^2 \geqslant 1 - \frac{1}{n^2}. \tag{8}$$

It is well known that for a coherent state $\langle \Delta^2\hat{x}\rangle = \langle \Delta^2\hat{y}\rangle = 1/4$. Condition (7) means that $\langle \Delta^2\hat{x}\rangle_0$ is narrowed by a factor of 2n compared to that of a coherent state. Then, by virtue of the uncertainty principle, $\langle \Delta^2\hat{y}\rangle_0$ is increased by a factor of 2n. These results can be illustrated by means of the phase plane. In classical oscillation theory the state of harmonic oscillator is described by a point on the phase plane moving in a circle. A coherent state differs in that the point is "smeared" over an area of $\Delta x\Delta y \simeq 1/4$ (Fig. 1). An example of a state satisfying conditions (7) is given in Fig. 2, and Fig. 3 shows a state which satisfies conditions (7) and (8).

2. The question naturally arises of how to measure the phase of a supercoherent state. It is clear from conditions (7) that for measuring the phase to an accuracy of $\Delta C \simeq 1/n$ (we will use the simplified notation $\Delta C \equiv \sqrt{\langle \Delta^2\hat{C}\rangle}$) it is necessary to rather rapidly measure the coordinate to an accuracy of $\Delta x = 1/2\sqrt{2n}$ when $\langle\hat{x}\rangle_0 \leqslant 1/2\sqrt{2n}$.

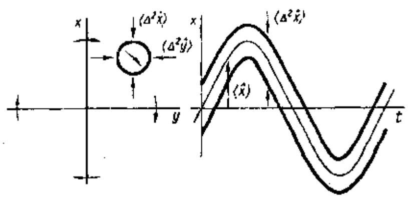

Fig. 1

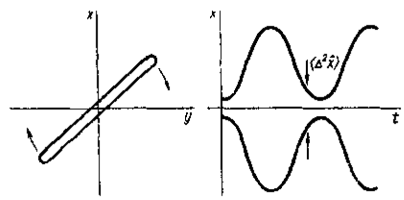

Fig. 2

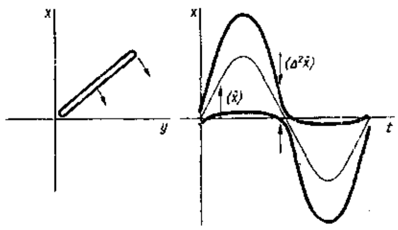

Fig. 3

We will assume that we have a displacement sender which resolves the coordinate to an accuracy of Δx, the detectable range of variations of the coordinate also being equal to Δx. In other words, this sender determines whether or not the object to be measured (in our case an oscillator) is located in the region $x_0 \pm \pm 1/2\ \Delta x$. It can therefore be called a "yes-no detector". We will also suppose that it is possible to assign the values of x_0 and Δx. Then our yes-no detector can be used to measure the phase of a supercoherent state. In fact, setting $\Delta x = \frac{1}{2\sqrt{\alpha}}$; $\alpha \gg 1$, $x_0 = x_p$ (x_p is the equilibrium position of the oscillator), we will obtain a device that will measure the phase of an accuracy of $\Delta C \approx 1/\alpha$. Of course, for this it is necessary to know x_p exactly, but, as was shown in Ref. [4], x_p can be determined to whatever accuracy is desired by preliminary measurement.

One can use a capacitance probe as a yes-no detector. It is known that if the pump frequency falls in the frozen-out band of the circuit (i.e., $\varkappa T_e \ll \hbar\omega_e$, where T_e and ω_e are the temperature and frequency of the circuit), a capacitance probe will permit the following resolution of the displacement in time τ [5]:

$$\Delta q = \Delta x \sqrt{\frac{\hbar}{2m\omega}} \simeq \frac{d}{Q_e} \sqrt{\frac{\hbar\omega_e R}{u_0^2 \tau}}, \tag{9}$$

where d is the distance between the plates of the capacitor, Q_e and R are the Q and resistance of the circuit, and u_0 is the amplitude of the pump generator. This introduces in accordance with the uncertainty principle a pertubation of the momentum of $\Delta y = \frac{1}{4\Delta x}$. If the pump frequency does not fall in the band of the circuit, the resolution of the displacement is worse by a factor of Q_e. One can therefore consider the detectable range of variations in the coordinate to be d/Q_e. In order to obtain a yes-no detector for measuring the phase to an accuracy of $\Delta C \leq 1/\alpha$, one must require (see Eq. (9)).

$$\frac{\hbar\omega_e R}{u_0^2 \tau} \simeq 1, \quad \frac{d}{Q_e} = \frac{1}{4}\sqrt{\frac{\hbar}{m\omega\alpha}},$$

and it is not hard to show that for this $\tau \gg 1/2\ \alpha\omega$.

Of course, the proposed procedure for measuring a supercoherent state can also be used to create one, since this procedure enables one to satisfy conditions (7). Here, it is true, one cannot in addition guarantee condition (8).

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Department of Oscillation Physics