THEORY OF INDUCED SYNCHROTRON RADIATION FOR THE ULTRARELATIVISTIC CASE

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Formulas derived in fundamental studies by A. A. Sokolov and I. M. Ternov on synchrotron radiation are used for the probabilities of induced transition, wave functions, and the electron energy spectrum in a magnetic field. The expression for the electron energy is used in deriving a formula for the radiation frequency, which is considerably higher than the nonrelativistic cyclotron frequency in the ultrarelativistic case. Products of matrix elements in the Dirac matrices are derived, and a formula is derived for the probabilities of induced emission and absorption on the assumption that the incident electromagnetic wave is linearly polarized. The values are substituted into the formula for the power in the stimulated transitions to derive an expression for the power in the induced emission for a relativistic rotating electron. The formula implies that relativistic electrons can be used to generate high-power electromagnetic radiation in the submillimeter and infrared wavelength ranges.

Relativistic electron beams provide promising sources of high-power radiation in the millimeter and submillimeter wavelength ranges. The theory of instruments that use relativistic electron beams has been developed mainly by classical methods and in part by numerical integration by computer. However, it has been pointed out [1] in relation to the theory of synchrotron radiation that quantum analytical methods are very useful. In our view, a quantum theory of induced synchrotron radiation is a logical development of fundamental researches on synchrotron radiation, particularly since such radiation is now used in sources of coherent electromagnetic radiation [1-3], since it gives a clear physical picture of the processes in devices with spiral electron beams (whether weakly relativistic or ultrarelativistic), and it provides an analogy with quantum generators for the optical UHF ranges.

Also, the method provides the valuable working formulas required for proper formulation of experiment.

We consider the induced emission from a relativistic electron moving along a spiral path in a constant and homogeneous magnetic field $\mathbf{H}=\{0,0,H\}$ and interacting with an external electromagnetic wave.

The following expression [1] gives the energy spectrum of the electron in that case:

$$W_n = c\hbar K_n = c\hbar\sqrt{k_0^2 + 4\gamma n + k_z}, \tag{1}$$

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where $k_0 = \frac{m_0 c}{\hbar}$, $\gamma = \frac{eH}{2c\hbar}$, k_z is the momentum of the electron along the field and n is the principle quantum number ($n=0, 1, 2, \ldots$). From (1) we get the expression for the natural frequencies of the rotating electron:

$$\omega_{n,n'} = \frac{W_n - W_{n'}}{\hbar}. \tag{2}$$

We substitute (1) into (2) and expand in terms of the ratio ν/n ($n \gg 1$, $\nu = n - n' \ll n$), which is equivalent to expansion in terms of Planck's constant $\hbar$. Everywhere we retain terms in ν/n not higher than the first. Then we get

$$\omega_{n,n-\nu} = \frac{\nu\omega_c}{1-\beta_\parallel \cos\theta}\left(1 + \frac{\nu}{4n}\beta_\perp^2 \frac{\sin^2\theta}{(1-\beta_\parallel\cos\theta)^2}\right), \tag{3}$$

where $\omega_c = \Omega\frac{m_0c^2}{W}$, $\Omega = \frac{eH}{m_0c}$ is the nonrelativistic cyclotron frequency, $c\beta_\perp$ and $c\beta_\parallel$ are components of the electron velocity perpendicular and parallel to the magnetic field, and θ is the angle of incidence of the electromagnetic wave in the spherical coordinate system. We have used the law of conservation of momentum $k_z - k_{z'} = \varkappa_{n,n'}\cos\theta$ in deriving (3).

Formula (3) with $\beta_\parallel = 0$ is the same as formula (7) of [4], while for $\beta_\parallel = 0$ and $\theta \sim \pi/2$ it is the same as (7) of [5].

The main interest attaches to the ultrarelativistic case $W \gg m_0c^2$ and $\beta_\perp^2 \ll \beta_\parallel^2$, $\beta_\parallel \sim 1$; the spontaneous emission from such an electron occurs at a small angle $\theta \sim \frac{m_0c^2}{W} \ll 1$, so we get $\cos\theta \simeq 1$, $\sin\theta \sim \theta \sim \frac{m_0c_2}{W}$. Also, $\beta_\perp^2 \ll \beta_\parallel^2$, so we can write $\beta_\parallel \simeq 1 - \frac{1}{2}\left(\frac{m_0c^2}{W}\right)^2$.

Then the expression for the natural frequency of the radiation in system (3) becomes

$$\omega_{n,n-\nu} = 2\Omega\nu\frac{W}{m_0c^2}\left(1 + 2\nu\frac{\hbar\Omega}{m_0c^2}\right). \tag{4}$$

We see from (4) that in the ultrarelativistic case $W \gg m_0c^2$ the natural radiation frequency $\omega_{n,n-\nu}$ will be considerably higher than the nonrelativistic cyclotron frequency. For instance, for $H = 50$ kOe $\nu = 1$, $\Omega \simeq 880$ GHz ($\lambda = 2{,}14$ mm), while for $W = 10$ MeV $\omega_{n,n-1} \simeq 57\,000$ GHz ($\lambda = 0{,}033$ mm), and for $W = 50$ MeV $\omega_{n,n-1} \simeq 289\,000$ GHz ($\lambda = 0{,}007$ mm), so the natural emission frequencies are shifted to the submillimeter and infrared ranges in this energy range, which means that an ultrarelativistic electron beam can be used to generate high-power electromagnetic radiation in those frequency ranges.

The general methods of quantum electrodynamics [3] give us the expression for the probability of induced transitions from state with energy W_n to state W_n' as

$$w_{n,n'} = \frac{2\pi e^2 c N(\varkappa)}{\hbar L^3 \varkappa}\left[(\bar{\boldsymbol{\alpha}}^+\bar{\boldsymbol{\alpha}}) - (\bar{\boldsymbol{\alpha}}^+\boldsymbol{\varkappa}^\circ)(\bar{\boldsymbol{\alpha}}\boldsymbol{\varkappa}^\circ)\right]\frac{\partial}{\partial t}\left|\int_0^t e^{-ict(\varkappa_{n,n'}-\varkappa)}\,dt\right|^2, \tag{5}$$

where $N(\varkappa)$ is the number of photons in the incident wave of frequency $\omega = c\varkappa$, and $\boldsymbol{\alpha}$ is a matrix element of the Dirac matrices, $\boldsymbol{\varkappa}^\circ = \boldsymbol{\varkappa}/|\boldsymbol{\varkappa}|$.

We assume that the external electromagnetic wave is linearly polarized and that the electric vector lies along the radius from the center of the orbit of the electron (the σ component), and further that there is damping in the system

caused by the finite time spent by the electron in the initial state (on the average equal to $\tau/2$), which is introduced by means of the truncating time factor $e^{-t/\tau}$ in the integral in (5). Then (5) takes the following form [3] for time intervals $t \gg \tau$

$$w_{n,n'} = \frac{2\pi c e^2 N(\varkappa)}{\hbar L^3 \varkappa} (\bar{\alpha}_1^+ \bar{\alpha}_1) \frac{2\tau}{\tau^2 (\omega_{n,n'} - \omega)^2 + 1}, \tag{6}$$

where $\bar{\alpha}_1$ is the matrix element the Dirac matrix α_1:

$$\bar{\alpha}_1 = \int \psi_{n'}^+ e^{-i\varkappa \mathbf{r}} \alpha_1 \psi_n d^3x. \tag{7}$$

We use the expression for the electron wave functions derived in [3] to get an explicit expression for $\bar{\alpha}_1$. We then sum over all the radial transitions to get the expression

$$\bar{\alpha}_1^+ \bar{\alpha}_1 = \frac{1}{2} \left[\frac{KK' - k_\theta^2 - k_z k_{z'}}{2KK'} (I_{n,n'-1}^2 + I_{n-1,n'}^2) - \frac{4\gamma \sqrt{nn'}}{KK'} \cdot I_{n,n'-1} \cdot I_{n-1,n'} \right], \tag{8}$$

where $K \equiv K_n$, $K' \equiv K_{n'}$, $I_{n,n'} = I_{n,n'}(y)$ are functions that are expressed in terms of generalized Laguerre polynomials and $y = \varkappa^2 \sin^2\theta / 4\gamma$.

We expand the coefficients in (8) as series in ν/n and retain only the linear terms to get

$$\bar{\alpha}_1^+ \bar{\alpha}_1 = \frac{\beta_\perp^2}{4} \left[1 - \frac{\nu}{2n} \left(1 \pm \frac{\beta_\perp^2}{1 - \beta_\parallel \cos\theta} \right) \right] R_{n,n\mp\nu}^2. \tag{9}$$

Here $R_{n,n'} = I_{n,n'-1}(y) - I_{n-1,n'}(y)$, where the minus sign corresponds to induced emission and the plus sign corresponds to induced absorption.

We can now write an expression for the power of the transitions in the system produced by the external elecgromagnetic field:

$$P_n = \hbar\omega (w_{n,n+\nu} - w_{n,n-\nu}). \tag{10}$$

Then (6) and (9) give (10) the form

$$P_n = \hbar\omega \frac{2\pi c e^2 N(\varkappa)}{\hbar L^3 \varkappa} \frac{\beta_\perp^2}{4} 2\tau n S_1, \tag{11}$$

where

$$S_1 = \frac{\left[1 + \frac{\nu}{2n} \left(1 - \frac{\beta_\perp^2}{1 - \beta_\parallel \cos\theta} \right) \right] R_{n,n+\nu}^2}{\tau^2 (\omega_{n,n+\nu} - \omega)^2 + 1} - \frac{\left[1 - \frac{\nu}{2n} \left(1 - \frac{\beta_\perp^2}{1 - \beta_\parallel \cos\theta} \right) \right] R_{n,n-\nu}^2}{\tau^2 (\omega_{n,n+\nu} - \omega)^2 + 1}. \tag{12}$$

We represent the denominator of (12) as

$$\tau(\omega_{n,n-\nu}-\omega)=\tau(\omega_{n,n+\nu}-\omega)+\tau(\omega_{n,n-\nu}-\omega_{n,n+\nu})\simeq x+\frac{\nu}{2n}\tau Q\beta_{\perp}^{2}, \tag{13}$$

where $Q=\nu\frac{\omega_c\sin^2\theta}{(1-\beta_{\parallel}\cos\theta)^2}$.

We also use the Bessel-function approximation for the Laguerre polynomials [6]:

$$I_{n,n'}(y)\simeq J_{\nu}(\sqrt{y}(\sqrt{n}+\sqrt{n'})).$$

Then with accuracy up to ν/n

$$R_{n,n\mp\nu}^{2}(y)=4J_{\nu}^{'2}(2\sqrt{yn})\left[1\mp\frac{\nu\sqrt{yn}}{n}\frac{J_{\nu}''(2\sqrt{yn})}{J_{\nu}'(2\sqrt{yn})}\right]. \tag{14}$$

We substitute (13) and (14) into (12) and use the fact that

$$z=2\sqrt{yn}=\frac{\omega}{\omega_c}\beta_{\perp}\sin\theta,$$

to get

$$S_1=\frac{4\nu J_{\nu}^{'2}(z)}{n(1+x^2)}\left[\frac{\nu^2-z^2}{z}\frac{J_{\nu}(z)}{J_{\nu}'(z)}-\frac{\beta_{\perp}^2}{1-\beta_{\parallel}\cos\theta}+\frac{\beta_{\perp}^2 xQ\tau}{1+x^2}\right]. \tag{15}$$

We transform the coefficient in (11) on the basis of the expression for the electromagnetic field energy density:

$$\frac{E^2}{4\pi}=\frac{\hbar c\varkappa N(\varkappa)}{L^3},$$

where E is the amplitude of the electric field in the wave.

Then the expression for P_n becomes

$$P_n=\frac{2e^2E^2c^2\tau}{W}\frac{\nu\omega_c}{\omega}S', \tag{16}$$

where

$$S=\frac{J_{\nu}^{'2}(z)}{1+x^2}\left[\frac{\nu^2-z^2}{z}\frac{J_{\nu}(z)}{J_{\nu}'(z)}-\frac{\beta_{\perp}^2}{1-\beta_{\parallel}\cos\theta}+\frac{\beta_{\perp}^2 xQ\tau}{1+x^2}\right];$$
$$z=\frac{\omega}{\omega_c}\beta_{\perp}\sin\theta,\quad Q=\frac{\nu\omega_c\sin^2\theta}{(1-\beta_{\parallel}\cos\theta)^2}. \tag{17}$$

This formula is correct for any value of ν and for any electron energy.

It is of interest to compare (16) and (17) with results obtained within the

framework of classical theory [7]. As would be expected, (16) and (17) agree well with the corresponding expressions of [7], although a certain difference in the third term of (17) is due to the different distribution of the natural emission frequencies. On the other hand, results from a quantum-mechanical consideration on the basis of the assumptions of [5] $\left(\beta_{\parallel}=0,\ \theta\sim\frac{\pi}{2},\ \omega\simeq\nu\omega_c\right)$ agree completely with (16) and (17) taken for the same conditions.

In the weakly relativistic case, (16) becomes Schneider's formula [8].

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