ESTIMATION OF NONLINEAR EFFECTS ATTENDANT TO VERTICAL PROPAGATION OF NONMONOCHROMATIC ACOUSTIC RADIATION IN THE ATMOSPHERE

S. N. Kulichkov

Vestnik Moskovskogo Universiteta. Fizika,
Vol. 36, No. 2, pp. 28-33, 1981

UDC 534.222.2:551.596.1

The generation of a difference harmonic attending the vertical propagation in the atmosphere of two acoustic waves of close frequency is considered. Pumping of energy from the difference to the primary harmonic is possible at a certain distance from the source. The energy of the difference harmonic, after formation of a discontinuity in the primary radiation, tends to some constant value, determined by the ratio of the difference and fundamental frequencies. It is found that nonlinear effects manifest themselves in acoustic waves only at Mach numbers higher than some constant quantity, which depends on the values of the Knudsen number for the atmosphere.

The effects attending the propagation of acoustic waves of finite amplitude in continuous media have been extensively explored in nonlinear acoustics [1-3]. The most characteristic nonlinear effects here are the formation of a shock wave, appearance of a constant component, and generation of a difference harmonic. These effects may manifest themselves perceptibly in a real atmosphere. The amplitude of vibrational velocity in vertically propagating acoustic waves increases with altitude as $\rho_0^{-1/2}$; in addition, the profile of this amplitude is perceptibly distorted. The half period when the velocity is negative becomes smaller than π, the half period when the velocity is positive widens [4]. Here the amplitude of the velocity at large distances from the source tends to some constant value, controlled by the ratio of the wavelength to the height of the homogeneous atmosphere [5]. However, as far as we know, generation of the difference harmonic is a phenomenon that has not been investigated in detail.

In the case of a homogeneous medium, when the generation of the difference harmonic is examined within the framework of the three-mode interaction, energy is found to move from the difference to the primary harmonics [1]. The purpose of the present study is to investigate the generation of the difference acoustic frequency in nonmonochromatic acoustic radiation, propagating vertically in the atmosphere, and also to make comparative analysis of varying accompanying effects, in order to clarify the optimum conditions for transformation of above-ground nonmonochromatic acoustic radiation into low-frequency radiation in the upper layers of the atmosphere.

The investigation will be performed in the approximation of three-wave interaction with allowance for appearance of discontinuities in the primary radiation.

1. The equation describing the propagation of acoustic waves vertically upward in the atmosphere is (see [5]):

$$\frac{\partial v}{\partial z}-\frac{\beta}{2}v-\frac{\varepsilon}{2c_0^2}\frac{\partial v^2}{\partial\tau}-\frac{\nu_0\exp\left\{\int\limits_0^z\beta\,dz\right\}}{2c_0^3}\frac{\partial^2 v}{\partial\tau^2}=0, \tag{1}$$

where v is the vertical velocity, z is the vertical coordinate, $\tau=t-z/C_0$; ν_0 is the molecular dissipation coefficient, $\beta=1/H$; H is the height of the homogeneous atmosphere.

We write Eq. (1) in nondimensional form:

$$\frac{\partial U}{\partial X}-2\varepsilon\frac{M_0}{\sigma}U\frac{\partial U}{\partial\Theta}-\frac{Kn}{\alpha^2}(X+1)\frac{\partial^2 U}{\partial\Theta^2}=0, \tag{2}$$

where

$$\Theta=\omega\tau;\ X=\exp\left\{\int\limits_0^z\frac{\beta}{2}dz\right\}-1,$$

$$v=U_0U\exp\left\{\int\limits_0^z\frac{\beta}{2}dz\right\};\quad \alpha=\frac{\lambda}{2\pi H};\quad M_0=\frac{U_0}{c_0}, \tag{3}$$

Kn = l_0/H is the Knudsen number, whereas $l_0=\nu_0/c_0$ is the molecular mean free path.

It is seen from Eqs. (2) and (3) that in the linear approximation, starting with some distance z, dissipation predominates over inhomogeneity, and one observes intensive damping, so that effects associated with density stratification become insignificant. In nondimensional coordinates this distance is:

$$(X+1)_{\text{damp}}=\frac{\alpha}{\sqrt{\text{Kn}}}. \tag{4}$$

We also note that Eq. (2) in the case of $\nu=\nu_0(X+1)$, is Burgers' equation and permits the formation of a shock wave in the primary radiation, starting with some distance $(X + 1)_{\text{dif}}$ (see [1]), which, in our notation, has the form

$$(X+1)_{\text{dif}}=\frac{\alpha}{2\varepsilon M_0}. \tag{5}$$

2. Let us consider the generation of the difference component in vertical propagation in the atmosphere of nonmonochromatic acoustic radiation to the distances of shock-wave formation $(X + 1)_{\text{dif}}$. As previously noted, Eq. (2) will be solved within the framework of the three-mode interaction, following [1]. We shall seek a solution for U in the form

$$U=A_1(X)\sin\Theta_1+A_2(X)\sin\Theta_2-A_3(X)\sin\Theta_3, \tag{6}$$

where

$$\Theta_i=\omega_i\tau;\ i=1,\ 2,\ 3;\ \omega_3=\omega_1-\omega_2. \tag{7}$$

Substituting Eq. (6) into (2) and rearranging, we obtain for resonance modes satisfying (7), the equations

$$\frac{d\widetilde{A}_1}{dX} = -b_1 \exp\left[\frac{m_1 - m_2 - m_3}{2}(X+1)^2\right]\widetilde{A}_2\widetilde{A}_3,$$
$$\frac{d\widetilde{A}_2}{dX} = b_2 \exp\left[\frac{m_2 - m_1 - m_3}{2}(X+1)^2\right]\widetilde{A}_1\widetilde{A}_3, \qquad (8)$$
$$\frac{d\widetilde{A}_3}{dX} = \frac{b_1 + b_2}{2} \exp\left[\frac{m_3 - m_1 - m_2}{2}(X+1)^2\right]\widetilde{A}_1\widetilde{A}_2,$$

where

$$A_{1,2} = \widetilde{A}_{1,2} \exp\left[-\frac{m_{1,2}}{2}(X+1)^2\right];$$
$$A_3 = \left(\frac{2\omega_3}{\omega_1+\omega_2}\right)^{1/2}\widetilde{A}_3 \exp\left[-\frac{m_3(X+1)^2}{2}\right], \qquad (9)$$
$$b_{1,2} = \frac{\varepsilon M_0}{\alpha_{1,2}}\left(\frac{2\omega_3}{\omega_1+\omega_2}\right)^{1/2}; \quad m_i = \frac{\mathrm{Kn}}{a_i^2}; \quad i = 1, 2, 3. \qquad (10)$$

An exact analytic solution of Eq. (8) cannot be obtained, and hence we shall solve it in the low-viscosity approximation:

$$m_{1,2}(X+1) \ll b_{1,2}. \qquad (11)$$

Using (11), Eq. (8) can be made to yield the following relationship between the different A_i:

$$\widetilde{A}_1^2 \simeq \exp\left[(m_1 - m_2)(X+1)^2\right] \times \left[1 - \frac{b_1}{b_2}(\widetilde{A}_2^2 - 1)\right],$$
$$\widetilde{A}_3^2 \simeq \exp[-(m_2 - m_3)(X+1)^2] \times [\widetilde{A}_2^2 - 1]\,\frac{b_1 + b_2}{2b_2}. \qquad (12)$$

It was assumed that amplitudes A_1 and A_2 of the primary waves are the same and equal to unity in nondimensional form.

Substituting (12) into the second of equations (8), and performing transformations as in [1], we obtain the following expressions for A_i, written by means of elliptical functions

$$A_1 = \exp\left[-\frac{m_2(X+1)^2}{2}\right]\mathrm{sn}\left[\mathrm{K}(k) - \frac{b_1+b_2}{\sqrt{2}}\,y\right],$$
$$A_2 = \sqrt{\frac{\omega_1+\omega_2}{\omega_1}}\exp\left[-\frac{m_2(X+1)^2}{2}\right]\mathrm{dn}\left[\mathrm{K}(k) - \frac{b_1+b_2}{\sqrt{2}}\,y\right], \qquad (13)$$
$$A_3 = \sqrt{\frac{\omega_3}{\omega_1}}\exp\left[-m_2\frac{(X+1)^2}{2}\right]\mathrm{cn}\left[\mathrm{K}(k) - \frac{b_1+b_2}{\sqrt{2}}\,y\right],$$

where

$$y = \int_0^X \exp\left[-\frac{m_2(X+1)^2}{2}\right]dX, \qquad (14)$$

$k^2 = b_2/(b_1+b_2)$; K (**k**) is an elliptical integral.

It is seen from Eq. (13) that, without allowance for discontinuities in the primary radiation, the amplitude of the difference harmonic can attain a maximum value at some distance from the source, and pumping of energy from the difference to the primary harmonics becomes possible. We have

$$(X+1)_{pr} = 1 + \frac{K(k)\,a_3}{\varepsilon M_0} \sqrt{\frac{\omega_3}{\omega_1+\omega_2}} \sim \sqrt{\frac{\omega_1}{\omega_3}}\,(X+1)_{dif}. \tag{15}$$

3. Let us consider the generation of the difference harmonic in the region following the formation of discontinuities in the primary radiation $(X + 1) \geq \geq (X + 1)_{dif}$.

It was noted above that in the case of $\nu \sim \nu_0(X+1)$ Eq. (2) has the form of the Burgers equation, the solution of which at $(X + 1)_{dif} < (X + 1)$ can be written similarly to Fay's solution [1]:

$$U = \frac{kl_0}{\varepsilon M_0} \sum_{n=1}^{\infty} \frac{\sin n\Theta}{\mathrm{sh}\left(\pi n \frac{L}{2}\right)}, \tag{16}$$

where

$$L = (1 + BX)\frac{2l_0}{\varepsilon M_0 \lambda}; \quad B = 1/X_{dif}.$$

It is seen from Eq. (16) that, in the presence of dissipation, the jumps in the amplitude of velocity in the vicinity of the shock front smooth out. Here the amplitude of the first harmonic changes with distance $1/\mathrm{sh}\left(\frac{\pi L}{2}\right)$. It will be assumed that, after formation of discontinuities, the amplitudes of the primary waves change primarily due to generation of their higher harmonics, and not to generation of the difference wave. In this approximation we seek the solution of Eq. (2) in the region after the formation of discontinuities in the primary radiation in the form

$$U = \frac{k_1 l_0}{\varepsilon M_0} \frac{\sin\Theta_1}{\mathrm{sh}\left(\frac{\pi L_1}{2}\right)} + \frac{k_2 l_0}{\varepsilon M_0} \frac{\sin\Theta_2}{\mathrm{sh}\left(\frac{\pi L_2}{2}\right)} - U_3(X)\sin\Theta_3. \tag{17}$$

It will be assumed for simplicity in mathematical presentation that the nature of the nonlinear distortions for the primary waves is approximately the same, i.e., we shall set in Eq. (17) $k_1 \simeq k_2 \simeq \bar{k} = (k_1+k_2)/2$. On this assumption we can neglect the damping of the difference harmonic due to molecular dissipation. Substituting Eq. (17) into (2) with allowance for the fact that $\Theta_3 = \Theta_1 - \Theta_2$, we obtain for the difference harmonic $U_3(X)$ at $(X + 1) > (X + 1)_{dif}$ the equation

$$\frac{\partial U_3}{\partial X} = \frac{\varepsilon M_0}{a_3}\left(\frac{kl_0}{\varepsilon M_0}\right)^2 \frac{1}{\mathrm{sh}^2\left[\frac{(1+BX)\,kl_0}{2\varepsilon M_0}\right]}, \tag{18}$$

$$U_3(X_p) = \frac{\sqrt{2}}{2}\,\frac{\omega_3}{\omega_1}. \tag{19}$$

The value of $U_3(X_{dif})$ was obtained from Eq. (13).

Integrating Eq. (18) with condition (19), we shall obtain the following expression for the amplitude of the difference harmonic at $(X + 1) > (X + 1)_{dif}$:

$$U_3(X) = \frac{\sqrt{2}}{2}\,\frac{\omega_3}{\omega_1} + \left(\frac{kl_0}{\varepsilon M_0}\right)\frac{a}{a_3}\left[\mathrm{cth}\left(\frac{kl_0}{\varepsilon M_0}\right) - \mathrm{cth}\left(\frac{kl_0}{a}X + \frac{kl_0}{2\varepsilon M_0}\right)\right]. \tag{20}$$

Converting to dimensional variables, we can obtain the vibrational velocity for the difference harmonic:

$$v_3 \cong U_0 \left(\frac{\sqrt{2}}{2} + \frac{a-1}{a+1}\right) \frac{\omega_3}{\omega_1} \exp\left\{\int_0^z \frac{dz}{2H}\right\} \sin \omega_3 \left(t - \frac{z}{c_0}\right). \tag{21}$$

where

$$a = \frac{X+1}{(X+1)_{\text{dif}}},$$

The energy $E(\Theta_3, X)$ of the difference harmonic is

$$E(\Theta_3, X) = \frac{\rho_0 v_3^2}{2} = \frac{\bar{\rho}_0}{2} \left(\frac{\omega_3}{\omega_1}\right)^2 \left(\frac{\sqrt{2}}{2} \quad \frac{a-1}{a+1}\right)^2 U_0^2.$$

Since $E(\Theta_{1,2}) = \bar{\rho}_0 U_0^2/2$, we obtain the following expression for coefficient ζ of transformation of the energy of primary radiation to that of the difference harmonic:

$$\zeta = \frac{E(\Theta_3, X)}{\sqrt{E(\Theta_1) E(\Theta_2)}} = \left(\frac{\omega_3}{\omega_1}\right)^2 \left(\frac{\sqrt{2}}{2} + \frac{a-1}{a+1}\right)^2. \tag{22}$$

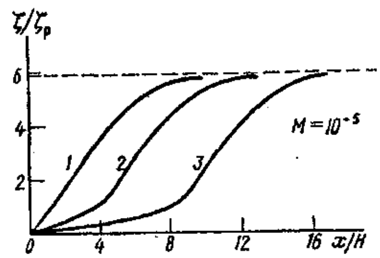

Fig. 1

A plot of ζ as a function of the distance to the source at $M_0 = 10^{-5}$ and $\lambda/H = 10^{-4}$, 10^{-3} and 10^{-2} (curves 1, 3, and 3) is shown in Fig. 1.

It is seen from Fig. 1 that the process of transformation of the energy of primary radiation into that of the difference harmonic attains saturation, and the amplitude of the latter harmonic tends to some constant value, which is a function only of the mismatch ω_3/ω_1. For example, the energy of the difference wave at a frequency of 4 Hz, generated upon interaction in the atmosphere of acoustic waves, generated by ground-based sources at frequencies of 42 and 38 Hz can amount, at an altidue of 80 km, approximately 3% of the energy of the ground-level radiation. In the case of a real atmosphere, when $\nu \sim \nu_0 (X+1)^2$, one naturally expects a slower rise in ζ with the altitude than in the case of $\nu \sim \nu_0 (X+1)$. The nondimensional width of the shock front is then given by the expression $L' \sim L(X+1)$ (see [5]). It can be shown that at $(X + 1) < (X + 1)_{\text{damp}}$ the deviations in nonlinear distortions for the real atmosphere from the case considered above are insignificant. The case of $\nu \sim \nu_0 (X+1)^2$ has been examined in detail by the present author in [6].

It can be found from Eqs. (16) and (20) that, starting with some distance $(X + 1)$, the amplitude of the difference radiation will be greater than the amplitude of the primary waves, and pumping of energy from the difference to the primary harmonics becomes possible. We have

$$(X+1)'_{\text{pr}} \sim \sqrt{\frac{\omega_1}{\omega_3}}\,(X+1)_{\text{pr}} \sim \frac{\omega_1}{\omega_3}(X+1)_{\text{dif}} = \frac{\Lambda/H}{4\pi e M_0}. \tag{23}$$

Figure 2 is a plot of $z/H = 2\ln(X + 1)'_{\text{pr}}$ vs. Λ/H at $M = 10^{-4}$ (1) and 10^{-5} (2); Λ is the length of the difference wave.

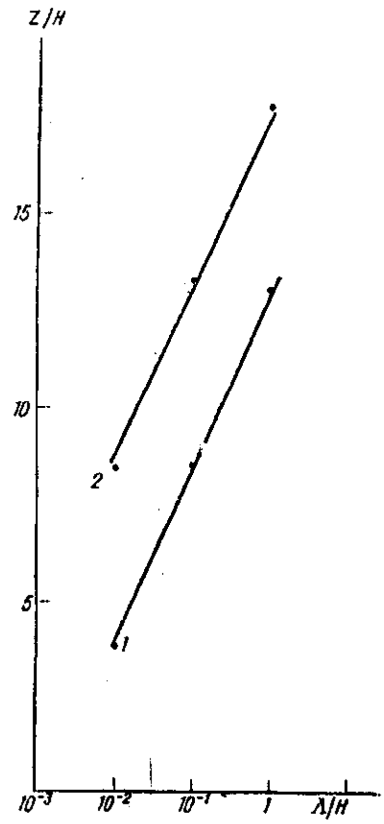

Fig. 2

As was mentioned above, Eq. (20) for amplitude U_3 of the difference harmonic is valid provided that $U_3 < U_{1,2}$. In this case it is still possible to use Fay's solution for the primary waves. Here the effects considered above have energy significance in the case when they manifest themselves up to the distance of damping of the primary radiation due to molecular dissipation $(X + 1)_{damp}$ (see the paper by the present author [6]). Substituting $(X + 1)_{damp}$ into Eqs. (17) and (20), we obtain a condition for the suitability of the solution under saturation conditions:

$$U_3 < U_{1,2} \quad \text{or} \quad M < \frac{\sqrt{\mathrm{Kn}}}{2\varepsilon}\,\frac{\omega_1}{\omega_3}. \qquad (24)$$

Nonlinear effects manifest themselves perceptibly when

$$(X + 1)_{dif} < (X + 1)_{damp} \quad \text{or} \quad M > \frac{\sqrt{\mathrm{Kn}}}{2\varepsilon}. \qquad (25)$$

We designate $\sqrt{\mathrm{Kn}}/2\varepsilon$ by M_{cr}. At $\mathrm{Kn} = 8\cdot10^{-12}$ it follows from Eqs. (25) that $M_{cr} = 1.175\cdot10^{-6}$. We note that the critical value of the Mach number is controlled only by the values of Kn and ε for the atmosphere and is independent of the wavelength.

REFERENCES

1. O. V. Rudenko and S. I. Soluyan, Theoretical Fundamentals of Nonlinear Acoustics [in Russian], Moscow, 1975.

2. K. S. Bakhvalov, Ya. M. Zhileikin, and O. V. Rudenko, "Concerning features of the operation of high-power parametric acoustic radiators," Akust. zhurnal., vol. 24, no. 1, pp. 125-127, 1978.

3. B. K. Novikov and O. V. Rudenko, "Generation of low-frequency harmonics in the field of a high-power amplitude-modulated wave," Akust. zhurnal, vol. 23, no. 5, pp. 797-804, 1977.

4. N. N. Romanova, "Vertical distribution of acoustic waves of arbitrary frequency in an isothermal atmosphere," Izv. AN SSSR, Fiz. atm. i okeana, vol. 11, no. 3, pp. 233-238, 1975.

5. N. N. Romanova, "Concerning the vertical propagation of short acoustic waves in a real atmosphere," Izv. AN SSSR. Fiz. atm. i okeana, vol. 6, no. 2, pp. 134-145, 1970.

6. S. N. Kulichkov, Estimate of Nonlinear Effects in Acoustic Waves Propagating Along a Sloping Path in the Atmosphere. Preprint of the Institute of Atmospheric Physics, Moscow, 1979.

19 March 1979 Department of Atmospheric Physics