INVESTIGATION OF THE STABILITY OF LINEAR AND LINEARIZED SYSTEMS WITH FORMULAS OF THE FREE PARAMETER OF THE ANALYTIC ROOT LOCUS METHOD

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The paper examines a graphical method for determining domains of stability with respect to a free parameter of systems described by the characteristic equations $\Phi_n(p) + \Psi K_m(p) = 0$. Unlike frequency methods, it employs simpler graphs on the real plane (ω^2, K). The method is advantageous for investigating the stability of higher-order and multicontour systems. Features of the application of the method to certain particular cases are examined. The results are illustrated by numerical examples. In the case of higher-order systems it is advantageous to perform the calculations by computer.

The question of stability of linear dynamic systems is fully solved by constructing the trajectories of the roots of the characteristic equation

$$\Phi_n(p) + K\Psi_m(p) = (a_0p^n + a_1p^{n-1} + \ldots + a_n) + K(b_0p^m + b_1p^{m-1} + \\ + \ldots + b_m) = 0, \tag{1}$$

where K is a free parameter $(n \geq m)$ [1].

At specified initial and limiting points of the trajectories of roots stability can be investigated without constructing trajectories of roots, but using the general properties of root loci [2].

This paper considers a simple graphical method for determining the domains of stability of systems described by characteristic equation (1), based on construction of curves of the free parameter [3], on the $[\omega^2, K(\delta, \omega^2)]$ plane for $\delta = 0$. This method, unlike frequency methods [4], based on the construction of parametric curves of $u(\omega)$ and $v(\omega)$ on the complex (u,v) plane, uses simpler graphs as a function of the square of the frequency on the real (ω^2, K) plane. It is advantageous for investigating the stability of higher-order systems and also of multicontour systems.

This method was suggested in analogous form by Becker [5], however, it was not associated with the analytical root locus method and has not been sufficiently exhaustively discussed.

1. **Determination of the Domains of Stability of Class [n;m] Systems on the (ω^2, K) Plane.** Substituting $p = \delta + j\omega$ into Eq. (1) and separating the real and imaginary parts, we obtain:

$$[\Phi_r(\delta, \omega^2) + K\Psi_r(\delta, \omega^2)] + j\omega[\Phi_j(\delta, \omega^2) + K\Psi_j(\delta, \omega^2)] = 0, \tag{2}$$

where, according to the first of the present authors with Teodorchik [1]

$$\Phi_2(\delta, \omega^2) = \Phi_n^{(0)}(\delta) - \frac{\omega^2}{2!}\Phi_n^{(2)}(\delta) + \dots,$$

$$\Psi_r(\delta, \omega^2) = \Psi_m^{(0)}(\delta) - \frac{\omega^2}{2!}\Psi_m^{(2)}(\delta) + \dots,$$

$$\Phi_j(\delta, \omega^2) = \Phi_n^{(1)}(\delta) - \frac{\omega^2}{3!}\Phi_n^{(3)}(\delta) + \dots,$$

$$\Psi_j(\delta, \omega^2) = \Psi_m^{(1)}(\delta) - \frac{\omega^2}{3!}\Psi_m^{(3)}(\delta) + \dots.$$

Here the parenthetical superscript gives the order of the derivative with respect to δ.

The formulas of the free parameter for $\omega \neq 0$ have the form:

$$-K_r = \frac{\Phi_r(\delta, \omega^2)}{\Psi_r(\delta, \omega^2)} = \frac{\Phi_n^{(0)}(\delta) - \frac{\omega^2}{2!}\Phi_n^{(2)}(\delta) + \dots}{\Psi_m^{(0)}(\delta) - \frac{\omega^2}{2!}\Psi_m^{(2)}(\delta) + \dots}, \tag{3}$$

$$-K_j = \frac{\Phi_j(\delta, \omega^2)}{\Psi_j(\delta, \omega^2)} = \frac{\Phi_n^{(1)}(\delta) - \frac{\omega^2}{3!}\Phi_n^{(3)}(\delta) + \dots}{\Psi_m^{(1)}(\delta) - \frac{\omega^2}{3!}\Psi_m^{(3)}(\delta) + \dots}, \tag{4}$$

here $K_r = K_j = K$, and for $\omega = 0$

$$-K = \frac{\Phi_n(\delta)}{\Psi_m(\delta)} = \frac{a_0\delta^n + a_1\delta^{n-1} + \dots + a_n}{b_0\delta^m + b_1\delta^{m-1} + \dots + b_m}. \tag{5}$$

Equations (3) and (4) can be used for graphical construction of a root locus. For this [3] curves of K_r and K_j are constructed on the (ω^2, K) plane at fixed values of $\delta = \delta_0$. The values of ω corresponding to the case of $K_r = K_j$ are determined from the intersection of these curves. The values of the free parameter on the real axis, as can be seen by comparing Eqs. (3) and (5), are found from the intersection between curve K_r and axis K.

To investigate stability we shall set $\delta = 0$ in Eqs. (3) and (4), and write them in terms of the coefficients of polynomials $\Phi_n(p)$ and $\Psi_m(p)$:

$$-K_r = \frac{\Phi_r}{\Psi_r} = \frac{a_n - \omega^2 a_{n-2} + \omega^4 a_{n-4} - \dots}{b_m - \omega^2 b_{m-2} + \omega^4 b_{m-4} - \dots}, \tag{6}$$

$$-K_j = \frac{\Phi_j}{\Psi_j} = \frac{a_{n-1} - \omega^2 a_{n-3} + \omega^4 a_{n-5} - \dots}{b_{m-1} - \omega^2 b_{m-3} + \omega^4 b_{m-5} - \dots} \tag{7}$$

or

$$\Phi_r + K_r\Psi_r = (a_n - \omega^2 a_{n-2} + \omega^4 a_{n-4} - \dots) + K_r(b_m - \omega^2 b_{m-2} + \omega^4 b_{m-4} - \dots) = 0, \tag{6a}$$

$$\Phi_j + K_j\Psi_j = (a_{n-1} - \omega^2 a_{n-3} + \omega^4 a_{n-5} - \dots) + K_j(b_{m-1} - \omega^2 b_{m-3} + \omega^4 b_{m-5} - \dots) = 0. \tag{7a}$$

Then one can determine the possible critical frequencies ω_{cr} and critical values of the free parameter K_{cr} from the intersection of curves K_r and K_j at $\delta = 0$, constructed on the (ω^2, K) plane.

We shall single out domains of values of K on the (ω^2, K) plane at which characteristic equation (1) describes the continuum of stable systems. Substituting $p = j\omega$ into Eq. (1) and fixing $K = K_0$, we obtain formulas of Mikhailov's characteristic curve [4]:

$$u(\omega) = (a_n - \omega^2 a_{n-2} + \omega^4 a_{n-4} - \dots) + K_0 (b_m - \omega^2 b_{m-2} + \omega^4 b_{m-4} - \dots)$$

and

$$v(\omega) = \omega\left[(a_{n-1} - \omega^2 a_{n-3} + \omega^4 a_{n-5} - \dots) + K_0 (b_{m-1} - \omega^2 b_{m-3} + \omega^4 b_{m-5} - \dots)\right].$$

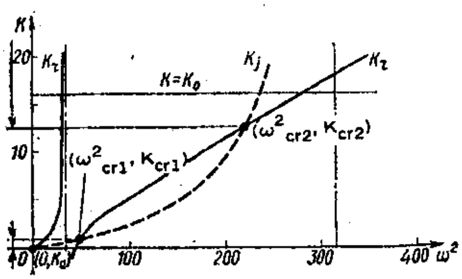

Fig. 1

It can be easily seen that for systems stable at some $K = K_0$ there should be satisfied a corollary of Mikhailov's criterion, according to which, when ω increases from 0 to $+\infty$ $(0<\omega<+\infty)$, functions $u(\omega)$ and $v(\omega)$ vanish sequentially $n - 1$ times. It follows from this that straight line $K = K_0$ on the (ω^2, K) plane should, for stable systems, sequentially intersect first curve K_r and then curve K_j, etc., $n - 1$ times.

The Vyshnegradskii condition which, for Eq. (1) has the form:

$$a_0>0,\ a_1>0, \dots, a_{n-m-1}>0,\ a_{n-m}+Kb_0>0, \dots, a_n+Kb_m>0. \qquad (8)$$

is always satisfied for stable systems. It imposes constraints on the value of K depending on the values of coefficients a_i and b_j. These equations will be used in graphical determination of the stability domains. If $a_n+Kb_m=0$, then $K = -a_n/b_m = K_a$ and the system is at the limit of aperiodic stability. This follows from Eq. (5) at $\delta = 0$. The intersection of curves K_r and axis K on the (ω^2, K) plane occurs in point $(0, K_a)$. In region K, where Eq. (8) is not satisfied, it is possible not to construct curves of K_r and K_j in order to reduce the volume of computations.

We shall explain the above by means of an example. Let us find the stability domain of system [6], the characteristic equation of which is:

$$(p^4+5{,}25p^3+43p^2)+K(p^3+17{,}01p^2+316{,}224p+588)=0. \qquad (9)$$

Substituting the numerical values of the coefficients of Eq. (9) into (6) and (7), we obtain formulas of the curves

$$K_r = \frac{43\omega^2-\omega^4}{588-17{,}01\omega^2} \quad \text{and} \quad K_j = \frac{5{,}25\omega^2}{316{,}224-\omega^2},$$

the graphs of which, as a function of the square of the frequency, are constructed in Fig. 1 at $K > 0$, where condition (8) is satisfied. The solid curves depict curve K_r, consisting of two branches: dashed - the upper branch of curve K_j (the

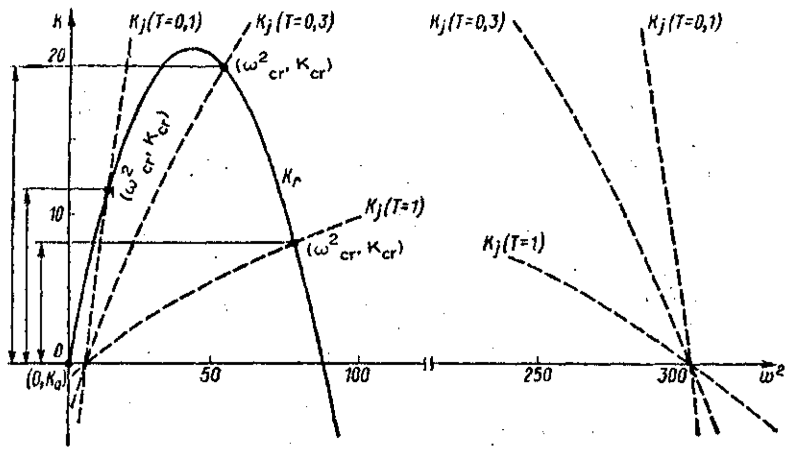

Fig. 2

lower branch is located in the region of negative values of K and hence is not shown) and dash-dotted lines - vertical asymptotes of curves K_r and K_j. The system is stable in regions $0 < K < 0.9$ and $12.5 < K < +\infty$, designated on the figure by arrows, since here any straight line $K = K_0$ intersects curves K_r and K_j three times in sequence. At $K_a = 0$ the system is at the limit of aperiodic stability, whereas at $K_{cr_1} = 0.9$ and $K_{cr_2} = 12.5$ - at the limit of vibrational stability with critical frequencies $\omega_{cr_1} = \sqrt{47} \simeq 6{,}9$ and $\omega_{cr2} = \sqrt{222} \simeq 14{,}9$.

The suggested method also makes it possible to investigate the effect of the parameters of the family of root loci at the stability domain with respect to the free parameter. As an illustration we consider a system described by characteristic equation [7]:

$$p(T_1p+1)(T_2p+1)(T_3T_4p^2+T_3p+1)+K(Tp+1)=0, \tag{10}$$

where $T_1 = 0.12$ sec, $T_2 = 0.77$ sec, $T_3 = 0.064$ sec, $T_4 = 0.085$ sec, T is the parameter of the family of the root loci, K is the free parameter.

Substituting numerical values into Eq. (10), we obtain

$$(0{,}000503\,p^5+0{,}0108\,p^4+0{,}155\,p^3+0{,}954\,p^2+p)+K(Tp+1)=0,$$

whence, using Eqs. (6) and (7), we arrive at the formulas of the curves

$$K_r = 0{,}954\omega^2 - 0{,}0108\omega^4 \quad \text{and} \quad K_j = \frac{-1+0{,}155\omega^2-0{,}000503\omega^4}{T}.$$

Figure 2 shows a curve of K_r and a family of curves K_j at T = 0.1 sec, 0.3 sec, and 1 sec, whence we find that the system is stable at $0 = K_a < K < K_{cr}$. It is seen that the critical value of the free parameter first increases with a rise in T and then, having attained a maximum, decreases.

2. **Determination of Stability Domains of Class [n,0] Systems.** For the particular case of $\Psi_m(p) = 1$ characteristic equation (1) becomes:

$$\Phi_n(p)+K=(a_0p^n+a_1p^{n-1}+...+a_n)+K=0. \quad (11)$$

Substituting m = 0 and b_0 = 1 into Eqs. (6a) and (7a), we obtain

$$-K_r=a_n-\omega^2a_{n-2}+\omega^4a_{n-4}-\ldots, \quad (12)$$

$$\Phi_j=a_{n-1}-\omega^2a_{n-3}+\omega^4a_{n-5}-\ldots=0, \quad (13)$$

where the last equation is an equation of the possible critical frequencies, and corresponds to curve K_j. It has a lower order than Eq. (11). Curves of K_j on the (ω^2, K) plane become vertical straight lines $\omega^2 = \omega^2_{cr}$, where ω^2_{cr} are positive real roots of Eq. (13).

Equations (8) simplify and can be written as:

$$a_0>0,\ a_1>0,\ \ldots,\ a_{n-1}>0,\ K>-a_n. \quad (14)$$

Let us find, as an illustration, the stability domains of system [7], the characteristic equation of which is

$$p(T_1p+1)(T_2p+1)(T_3p+1)(T_4T_5p^2+T_4p+1)+K=0, \quad (15)$$

where T_1 = 0.005 sec, T_2 = 0.12 sec, T_3 = 0.77 sec, T_4 = 0.064 sec, T_5 = 0.085 sec.

Substituting numerical values into Eq. (15) and removing the brackets, we obtain $(0{,}00000251\,p^6+0{,}000556\,p^5+0{,}0115\,p^4+0{,}16\,p^3+0{,}959\,p^2+p)+K=0$, whence, according to Eqs. (12) and (13)

$$K_r=0{,}959\,\omega^2-0{,}0115\,\omega^4+0{,}00000251\,\omega^6,$$
$$\Phi_j=1-0{,}16\,\omega^2+0{,}000556\,\omega^4=0.$$

We use the last equation to find the possible critical frequencies $\omega_{cr1}=\sqrt{6{,}4}\simeq2{,}5$ and $\omega_{cr2}=\sqrt{280{,}4}\simeq16{,}7$. By virtue of (14), the system is unstable at K < 0, hence it is possible not to consturct curve K_r here. As is seen from Fig. 3, the system is stable in region 0 < K < 5.7. At K_α = 0 the system is at the boundary of aperiodic stability, whereas at K_{cr1} = 5.7 it is at the limit of vibrational stability. Hence only ω_{cr1} is the critical frequency.

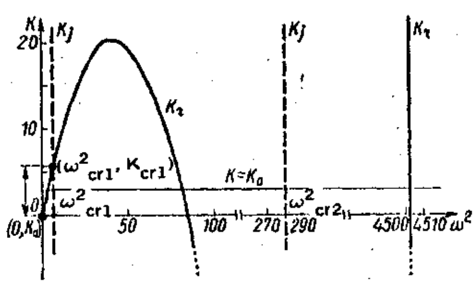

Fig. 3

3. Particular Cases of Determination of Domains of Stability. At certain values of coefficients a_i and b_j of characteristic equation (1) there arise particular cases of determination of domains of stability. They correspond to certain characteristic locations of the initial and limiting points of the root trajectories.

A) Let characteristic equation (1) have the form:

$$(a_0p^n+a_1p^{n-1}+\ldots+a_n)+Kp^m=0, \quad (16)$$

where m > 0 (m is the multiple limiting point at the coordinate origin of the p plane).

Substituting $b_0=1,\ b_1=b_2=\ldots=b_m=0,$ into Eqs. (6a) and (7a), we obtain for even m:

$$K_r = \frac{a_n - \omega^2 a_{n-2} + \omega^4 a_{n-4} - \ldots}{(-\omega^2)^{m/2}},\quad \Phi_j = a_{n-1} - \omega^2 a_{n-3} + \omega^4 a_{n-5} - \ldots = 0,$$

and for odd m:

$$\Phi_r = a_n - \omega^2 a_{n-2} + \omega^4 a_{n-4} - \ldots = 0,\quad -K_j = \frac{a_{n-1} - \omega^2 a_{n-3} + \omega^4 a_{n-5} - \ldots}{(-\omega^2)^{(m-1)/2}}.$$

the subscript r or j in equations of possible critical frequencies $\Phi_r = 0$ or $\Phi_j = 0$ shows to which curves, K_r or K_j, they correspond.

B) Let characteristic equation (1) have the form

$$(a_0p^n + a_1p^{n-1} + \ldots + a_n) + K[a_0p^n - a_1p^{n-1} + \ldots + (-1)^n a_n] = 0. \tag{17}$$

As can be easily seen, the limiting points of Eq. (17) are mirror images of the initial points relative to the imaginary axis $j\omega$.

Substituting $b_0=a_0,\ b_1=-a_1, \ldots, b_n=(-1)^n a_n$ into Eqs. (6a) and (7a), we obtain formulas of curves of K_r and K_j:

$$\Phi_r = a_n - \omega^2 a_{n-2} + \omega^4 a_{n-4} - \ldots = 0,\quad K_r = (-1)^{n+1} \tag{18}$$

and

$$\Phi_j = a_{n-1} - \omega^2 a_{n-3} + \omega^4 a_{n-5} - \ldots = 0,\quad K_j = (-1)^n, \tag{19}$$

which, on the (ω^2, K) plane, will have the form of vertical and horizontal lines. Subscripts r and j in Eqs. (18) and (19) show to which curves they correspond.

In both cases the stability domains are determined in the same manner as this was done for class [n,0] systems.

We note in conclusion that the above method of determination of domains of stability from formulas of the free parameter can be easily extended to the case when it is required to find the domains of the values of the free parameter, at which the system has a specified stability margin S. For this curves of K_r and K_j on the (ω^2, K) plane are constructed from Eqs. (3) and (4) at $\delta = -S$.

The formulas of curves K_r and K_j are suitable for computer calculation. They are, for example, simpler than the corresponding formulas of D partitioning with respect to one parameter [8].

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