CONCERNING NONLINEAR SYSTEMS, STABLE IN THE GURVITS SECTOR

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Nonlinear automatic control systems, obeying the Popov criterion of unconditional stability of the equilibrium position in sector [0,k] are discussed. The root loci technique is used for obtaining conditions, in general form, at which this sector is at maximum and equal to the Gurvits sector of corresponding linear systems. Systems with class [3,m] and [4,m] linear parts are investigated.

A nonlinear system obeying the Popov criterion [1] can be subdivided into a linear part and a nonlinear element with characteristic shown in Fig. 1. The graph of this characteristic lies in a sector delimited by straight lines $y = 0$ and $y = k\sigma$, or, using the standard terminology, in sector [0,k]:

$$0 \leqslant f(\sigma) \leqslant k\sigma. \tag{1}$$

The linear part of the system is described by a transfer function, which is written as [2]:

$$W(p) = \frac{\Psi_m(p)}{\Phi_n(p)}, \quad p = \delta + j\omega, \quad m \leqslant n; \tag{2}$$

$$\Psi_m(p) = b_0 p^m + b_1 p^{m-1} + \ldots + b_{m-1} p + b_m;$$
$$\Phi_n(p) = a_0 p^n + a_1 p^{n-1} + \ldots + a_{n-1} p + a_n. \tag{3}$$

The characteristic equation of the complete linear system (which is termed a class [n,m] system) with transfer function (2) is

$$\Phi_n(p) + K\Psi_m(p) = 0. \tag{4}$$

Since in the basic formulation of the Popov criterion one considers only negative feedback, we will treat gain factor K as positive. We assume the linear part to be stable over some range of values of K, starting with zero and ending with the critical K_{cr}, corresponding to transfer of roots from the stable to the unstable domain on the "p" plane of the root locus of Eq. (4). Loss of stability by the system may occur vibrationally (at $\omega_{cr} \neq 0$) or aperiodically (at $\omega_{cr} = 0$).

Let us consider for illustration the root locus of system [6,0] with a sextuple initial point (Fig. 2). Negative feedback has corresponding to it odd loci designated by a single arrow, whereas positive feedback has corresponding to it

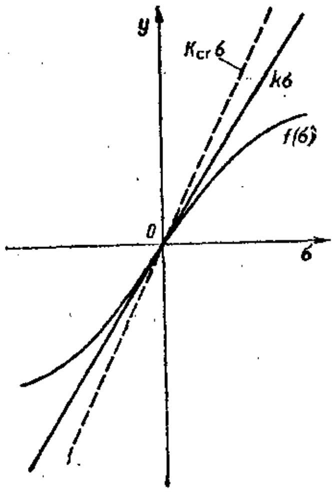

Fig. 1

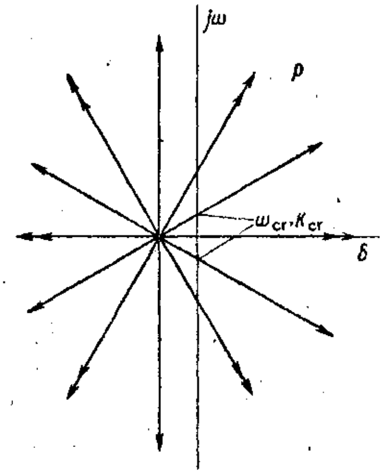

Fig. 2

even loci, designed by two arrows. Along the odd locus the system is transferred in an oscillatory manner from the left (stable) part of plane "p" to the right (unstable) part at $\pm j\omega_{cr}$ and $K = K_{cr}$. Interval $[0, K_{cr}]$ is termed the Gurvits interval, "angle" or "sector" since the Gurvits stability conditions for a linear system are satisfied over this interval. Certain linear systems may have several Gurvits intervals (for example, the class [9,0] system or conditionally stable systems). In this case one takes as K_{cr} the smallest of positive values of $K_{cr\ i}$, i.e., K_{cr} corresponds to the first intersection of the first odd trajectory with the imaginary axis. We note that in order for a system to be stable over interval $[0, K_{cr}]$, all its initial points should be situated in the left-hand side of the "p" plane, i.e., coefficients a_i of polynomial $\Phi_n(p)$ (the roots of which define the initial points) are all positive:

$$a_i > 0. \tag{5}$$

The Popov criterion employs the concept of modified frequency characteristic $W^*(\omega^2)$. Similarly to the ordinary frequency characteristic [3], we represent the real and imaginary parts of characteristic W^* in the form of ratios of polynomials:

$$W^*(x) = \frac{A(x)}{B(x)} + j\frac{C(x)}{B(x)}, \quad x = \omega^2,$$

where polynomials A, B and C have the form

$$A(x) = \sum_{i=0}^{l} A_{2i}x^i, \quad B(x) = \sum_{i=0}^{n} B_{2i}x^i, \quad C(x) = \sum_{i=0}^{r} C_{2i+1}x^{i+1}.$$

The coefficients of these polynomials are expressed in terms of the coefficients of characteristic equation (4) using the expressions given by the first of the present authors with his coworkers [3]:

$$A_{2i} = (-1)^i \sum_{j=0}^{2i} (-1)^j a_{n-j} b_{m-2i+j}; \tag{6}$$

$$C_{2i+1} = (-1)^i \sum_{j=0}^{2i+1} (-1)^j a_{n-j} b_{m+j-(2i+1)};$$

$$B_{2i} = (-1)^i \left\{ \sum_{j=0}^{j<i} 2\left[(-1)^{2i-j} a_{n-j} a_{n-2i+j}\right] + (-1)^i a_{n-i}^2 \right\}; \qquad (6)$$

$$l = \begin{cases} \frac{n+m}{2}, & (n+m) \text{ even} \\ \frac{n+m-1}{2}, & (n+m) \text{ odd,} \end{cases} \qquad r = \begin{cases} \frac{n+m-2}{2}, & (n+m) \text{ even} \\ \frac{n+m-1}{2}, & (n+m) \text{ odd.} \end{cases} \qquad (7)$$

Using the above polynomial representation of W*, we now write Popov's conditions for a system, unconditionally stable in sector [0,k], in the form [4]

$$A(x) - qC(x) + \frac{1}{k} B(x) > 0, \ \forall x \geqslant 0. \qquad (8)$$

Here q and k are real numbers; k being nonnegative. In the general case $0 \leq k \leq K_{cr}$ (see Fig. 1). If the value of $k = K_{cr}$ is possible for a nonlinear system, then the corresponding nonlinear system, as its linear part, will be stable in the $[0,K_{cr}]$ Gurvits sector.

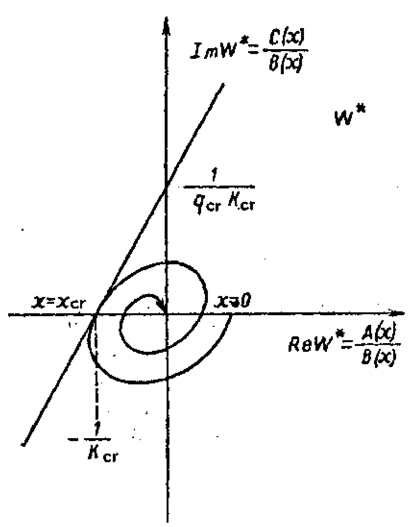

Fig. 3

The class of nonlinear systems stable in the Gurvits sector is rather wide. It is hence of interest to point out a criterion for the belonging of nonlinear systems to this class. The maximum stability sector for such systems is determined in an elementary manner from linear criteria, and this makes it unnecessary to resort to constructions or analytic studies, which are frequently labor-consuming.

Let us consider the modified frequency characteristic W*(x) of a system, stable in the Gurvits sector (Fig. 3). This should be a smooth, monotonic curve, such that the Popov straight line be tangent to it in the extreme left point of intersection between W* and the real axis A(x)/B(x). It is precisely this point which has corresponding to it the double root of Eq. (8) at $k = K_{cr}$. The slope of this Popov line, passing through point $(-1/K_{cr},0)$ and tangent to curve W* is determined by the "critical" angular coefficient $1/q_{cr}$. Accordingly, parameters q and k in Eqs. (8) for a "Gurvits" system are determinable.

The points of intersection between W* and the real axis (see Fig. 3) correspond to roots of the equation of critical frequencies of the linear system

$$C(x) = 0. \qquad (9)$$

The left extreme point which is of interest here corresponds either to zero or to the least positive root of this equation. The linear theory of root loci [2] or directly Fig. 3 can be used for obtaining an expression for the critical value of the gain factor in this point:

$$K_{cr} = -\frac{B(x)}{A(x)}\Big|_{x=x_{cr}}. \quad (10)$$

Let $x_{cr} = 0$. Then, from Eqs. (6) and (7)

$$K_{cr0} = -\frac{B(0)}{A(0)} = -\frac{a_n}{b_m}. \quad (11)$$

Using inequality (5), we obtain that, for negative feedback, aperiodic (at x_{cr} = = 0) transition through the imaginary axis of the "p" plane by a linear system is possible only if $b_m < 0$. Hence, if in polynomial $\Psi_m(p)$, defining the limiting points (see Eqs. (3), (4)), $b_m > 0$, then the value of K_{cr} is found from Eq. (10) at $x_{cr} \neq 0$; if, however, $b_m < 0$, then $[0, K_{cr0}]$, obtained from Eq. (11), will be the Gurvits sector of the system.

With reference to the above, we now formulate our stability criterion.

In order for a nonlinear system to be stable in the Gurvits sector, it is necessary and sufficient to satisfy the equations

$$A(x) - q_{cr}C(x) + \frac{1}{K_{cr}} B(x) \geqslant 0, \ \forall x \geqslant 0, \quad (12)$$

where

$$q_{cr} = \frac{[A(x)/B(x)]'_{x=x_{cr}}}{[C(x)/B(x)]'_{x=x_{cr}}} = \left[\frac{A'(x)}{C'(x)} + \frac{1}{K_{cr}} \frac{B'(x)}{C'(x)}\right]_{x=x_{cr}}; \quad (13)$$

The values of K_{cr} and x_{cr} are obtained from Eqs. (9)-(11), and the polynomials from Eqs. (6) and (7).

In general, conditions (12) are satisfied when the polynomial in the left-hand side of these equations is positive at $x \to +\infty$, and if it does not have positive real roots, with the exception of identical roots of even multiplicity [5]. In the case at hand, the absence of different positive real roots yields positiveness of the polynomial as $x \to +\infty$. In fact, the absence of different positive real roots here means that the straight line drawn through the extreme left point of intersection of curve W* with the real axis is only tangent to it at this, and possible, certain other points, but does not intersect this curve. Consequently, the entire curve W* lies to the right of this tangent and the polynomial in the left-hand side of Eqs. (12) is positive as $x \to +\infty$.

This means that the problem consists in investigating the positive real roots of Eq. (12). If there are no different roots, or identical roots of odd multiplicity among them, or if Eq. (12) does not at all have positive real roots, then the system under study is stable in the Gurvits sector.

Conditions (12) can be used for investigating the stability of systems of as high as desired orders in the Gurvits sector. On the other hand, for systems of not too high orders these conditions, additionally, make it possible to identify stability domains in the parameter space. This will be shown on an example of systems with linear part of class [3,m] and [4,m]. Subsequently, we shall refer for brevity to class [n,m] nonlinear systems, with the understanding that they have a class [n,m] linear part.

I. For class [3,m] systems Eq. (12) is a third-order equation:

$$f_0x^3+f_1x^2+f_2x+f_3=0. \tag{14}$$

Coefficients f_i are found directly from the coefficients of polynomials A, B, and C, using x_{cr} and K_{cr}. Let two identical roots be $x_1 = x_2 = x_{cr}$, and x_3 be the third root of Eq. (14), then

$$x_3=-f_3/f_0, \text{ if } x_{cr}\neq 0 \text{ (i.e. if } b_m>0),$$

and

$$x_3=-f_1/f_0, \text{ if } x_{cr}=0 \text{ (i.e. if } b_m<0).$$

It follows from positiveness of polynomial $f(x)$ at $x \to +\infty$ that $f_0 > 0$, and conditions of "Gurvits nature" of class [3,m] systems are written as:

$$f_3>0, \text{ if } x_{cr}\neq 0 \text{ (i.e. if } b_m>0); \tag{15}$$

$$f_1>0, \text{ if } x_{cr}=0 \text{ (i.e. if } b_m<0). \tag{16}$$

1. $b_m > 0$. We calculate $f_3 = a_3b_m + a^2{}_3/K_{cr}$. Obviously, $f_3 > 0$ for all class [3,m] systems. This means that all the class [3,m] systems are stable in the Gurvits sector, if $b_m > 0$, or, in other words, if the first transition through the imaginary axis at negative feedback for the linear system occurs in an oscillatory manner.

In the case of class [3,0], [3,1] and [3,2] systems the satisfaction of condition $b_m > 0$ yields graphic results, namely: systems which are stable in the Gurvits sector are of the following classes: 1) [3,0], 2) [3,1], which have a limiting point in the left-hand side of the "p" plane; 3) [3,2], if both limiting points lie either to the right or left in the "p" plane.

2. $b_m < 0$. We calculate coefficient f_1 in Eq. (14) and, according to Eq. (16), obtain conditions for the "Gurvits nature" of systems [3,m] at $x_{cr} = 0$:

$$\begin{aligned}
&[3;\ 1]\quad -a_0b_0+q_{cr0}(a_1b_0-a_0b_1)+\frac{a_1^2-2a_2a_0}{K_{cr0}}>0;\\
&[3;\ 2]\quad (a_1b_0-a_0b_1)+q_{cr0}(-a_2b_0+a_1b_1-a_0b_2)+\frac{a_1^2-2a_2a_0}{K_{cr0}}>0;\\
&[3;\ 3]\quad (-a_0b_0+a_1b_1-a_0b_2)+q_{cr0}(a_3b_0-a_2b_1+a_1b_2-a_0b_3)+\\
&\qquad\qquad +\frac{a_1^2-2a_2a_0}{K_{cr0}}>0.
\end{aligned} \tag{17}$$

II. For class [4,m] systems Eq. (12) is a fourth degree equation:

$$f_0x^4+f_1x^3+f_2x^2+f_3x+f_4=0. \tag{18}$$

At $x_{cr} \neq 0$ we write this equation in the form

$$(x-x_{cr})^2(ax^2+bx+c)=0, \tag{19}$$

where $a = f_0$; $b = 2f_0x_{cr} + f_1$ and $c = f_4/x^2{}_{cr}$. Obviously, in order for different positive real roots to be absent, we should have $a > 0$, $b > 0$ and $c > 0$. Since $f_0 > 0$ always, we obtain conditions for the belonging of class [4,m] systems to those stable in the Gurvits sector:

$$f_1 > 0,\ f_4 > 0;\ x_{cr} \neq 0. \tag{20}$$

If $x_{cr} = 0$, the order of Eq. (18) decreases by two, and Eqs. (20) are replaced by:

$$f_1 > 0,\ f_2 > 0;\ x_{cr} = 0. \tag{21}$$

1. $b_m > 0$. Coefficient $f_4 > 0$, since $f_4 = a_4 b_m + a_2^4/K_{cr}$. Stability in the Gurvits sector depends on satisfaction of the condition $f_1 > 0$:

$$[4;\ 0]\ a_1^2 - 2a_2a_0 > 0. \quad [4;\ 1]\ \frac{a_1^2 - 2a_2a_0}{K_{cr}} - a_0b_0q_{cr} > 0.$$

$$[4;\ 2]\ \frac{a_1^2 - 2a_2a_0}{K_{cr}} - (a_0b_1 - a_1b_0)\,q_{cr} - a_0b_0 > 0.$$

$$[4;\ 3]\ \frac{a_1^2 - 2a_2a_0}{K_{cr}} - (a_0b_1 - a_1b_0) - (a_2b_0 - a_1b_1 + a_0b_2)\,q_{cr} > 0. \tag{22}$$

$$[4;\ 4]\ \frac{a_1^2 - 2a_2a_0}{K_{cr}} - (a_2b_0 - a_1b_1 + a_0b_2) - (-a_3b_0 + a_2b_1 - a_1b_2 + a_0b_3)\,q_{cr} > 0.$$

We note that, for systems with real initial points, difference $a_1^2 - 2a_2a_0$ in inequalities (22) is a positive definite quadratic form. Consequently, at least all the class [4,0] systems with real initial points are stable in the Gurvits sector.

2. $b_m < 0$. This case corresponds to $x_{cr} = 0$. Solution of inequalities (21) in terms of coefficients of characteristic equation (4) yields complex expressions. Their general analysis is difficult, but specific systems are easily investigated using conditions (21).

Conclusions. The article formulates a criterion for determining the belonging of a nonlinear system to a class of systems stable in the Gurvits sector. For higher-order systems the study reduces to analysis of positive real roots of polynomial equation (12) with real roots. Stability domains in the Gurvits sector in the parameter space were obtained for class [3,m] and [4,m] systems. It is shown that all systems with linear part of class [3,0], and also [3,1], [3,2] and [3,3] are positive in the Gurvits sector when coefficients b_1, b_2 and b_3, respectively, are positive. In addition, systems with class [4,0] linear parts are stable in the Gurvits sector provided that the initial points of the linear part are real.

REFERENCES

1. M. A. Aizerman and F. R. Gantmakher, Unconditional Stability of Controlled Systems [in Russian], Moscow, 1963.
2. G. A. Bendrikov and K. F. Teodorchik, Root Loci of Linear Automatic Systems [in Russian], Moscow, 1964.
3. G. A. Bendrikov and U. Fonseka Araukho, "Frequency stability criteria

and their relationship to the root trajectories technique," Vestn. Mosk. un-ta, Ser. Fiz. Astron. [Moscow University Physics Bulletin], no. 2, pp. 166-176, 1973.

4. G. A. Bendrikov and G. A. Sidorova, "Application of the root locus technique for investigating the unconditional stability of nonlinear systems according to the Popov criterion," Vestn. Mosk. un-ta, Ser. Fiz. Astron. [Moscow University Physics Bulletin], no. 2, pp. 177-186, 1976.

5. D. D. Siljak, Nonlinear Systems, John Wiley and Sons, Inc., New York, London, 611 pp., 1968.

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