THE REGULAR MINIMUM OF THE SCALE FACTOR OF THE CO-MOVING SPACE IN EINSTEIN'S GRAVITATION THEORY

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The existence of an analytic solution of Einstein's equations, which contains a maximum number of physically arbitrary functions, in which scale factor R of the co-moving space passes through a regular minimum, corresponding to the finite density, is shown. Cases of an ideal and a viscous medium upon complete specification of the law of viscosity are considered separately. The nature of the distortion of the co-moving space, variation in the volume and density of the ideal and viscous fluids are examined near the regular minimum of R.

1. **Statement of the Problem.** The dropping of the requirement of homogeneity and isotropy of space in Einstein's gravitation theory makes it possible for a volume of an element of the co-moving space to pass through a regular finite minimum not only when the cosmological constant is $\Lambda > 0$, as in the case of homogeneous isotropic models, but also when $\Lambda < 0$ and $\Lambda = 0$ [1]. This means that the theory of an anisotropic inhomogeneous universe permits solutions of Einstein's equations according to which expansion replaces compression; here volume V of an element of the co-moving space passes through a regular minimum, corresponding to finite density. Consequently, the idea, according to which the observed expansion of the Metagalactic started from states of infinite density, is no longer inevitable. In order that a solution with a regular finite minimum of V be capable of describing the universe, it has to be stable, i.e., it has to belong to the general solution [2] (solution, containing a maximum number of physically arbitrary functions). In the opposite case a small change in the parameters of the solution may result in vanishing of the regular finite minimum of V. It is hence important to establish whether the volume of an element of the co-moving space in the general solution of Einstein's equations passes through a regular finite minimum. The purpose of the present article is to examine this problem for different structures of the energy-momentum tensor of the medium.

Zel'manov [1] proved the existence of a solution of Einstein's equations at any initial conditions, under which the scale factor of the co-moving space passes through a regular finite minimum. This result was obtained in that version of the semiinverse method, in which is ensured the satisfaction of signature conditions everywhere, but the viscosity law is not fully specified; as a result, the terms describing dissipative processes may become, with time, insufficiently small for the concept of viscosity to be applicable.

We note that the passing of a volume of an element of the co-moving space through a regular finite minimum even in the general solution of Einstein's equations does not contradict the conclusion on the existence of a general solution with a singularity, the latter consisting of a state of infinite density and infinite invariants of the four-dimensional curvature tensor, since the concept of a general solution for nonlinear partial differential equations, which the Einstein equations are, is not unique - there can exist more than one general solution (see the paper by Belinskii, et al. [2]). In principle there can also exist a general solution which has both a regular finite minimum of V and a singularity, which occur in different space-time domains.

Instead of the change in volume V of an element of the medium we shall consider the variation in quantity R, which is a generalized scale factor, proportional to the cubic root of V and defined, to within an arbitrary function of three-dimensional coordinates V_0: $R=(V/V_0)^{1/3}$, $V_0>0$, $\frac{\partial V_0}{\partial t}=0$. We also stipulate that all the coordinates x^α are real, $x^0=ct$, $ds^2=g_{\mu\nu}dx^\mu dx^\nu$, in the local Gallilean coordinate system $ds^2=(dx^0)^2-(dx^1)^2-(dx^2)^2-(dx^3)^2$.

We shall consider the matter as a medium free of nongravitational and noninertial mass forces, and shall neglect the energy flux relative to it, and the second viscosity.

We shall use equations of the law of gravitation

$$G_{0\nu}=-\varkappa\left(T_{0\nu}-\frac{1}{2}g_{0\nu}T\right)+\Lambda g_{0\nu}, \qquad (1)$$

$$G_{ik}=-\varkappa\left(T_{ik}-\frac{1}{2}g_{ik}T\right)+\Lambda g_{ik} \qquad (2)$$

and equations of the law of energy and momentum in the chronometrically invariant form with allowance for absence of the second viscosity and energy fluxes in the reference frame co-moving with the medium [3]:

$$\frac{{}^*\partial\rho}{\partial t}+3\frac{{}^*\dot{R}}{R}\left(\rho+\frac{p}{c^2}\right)=\frac{1}{c^2}\varepsilon_{jl}\Pi^{jl}, \qquad (3)$$

$$\frac{{}^*\partial p}{\partial x^i}-F_i\left(\rho+\frac{p}{c^2}\right)=\left({}^*\nabla_j-\frac{1}{c^2}F_j\right)\varepsilon^j_i. \qquad (4)$$

Here $G_{\mu\nu}$ is the four-dimensional Ricci tensor, $G=G^\nu_\nu$, $\varkappa$ is Einstein's gravity constant, c is the fundamental velocity, ρ is the chronometrically invariant mass density, p is the actual pressure, F_i is the chronometrically invariant vector of the gravitation-inertial force, $\Pi_{ik}=D_{ik}-Dh_{ik}/3$, D_{ik} is the chronometrically invaraint tensor of deformation rates, $D=D^i_i=\frac{{}^*\partial\ln\sqrt{h}}{\partial t}=3\,{}^*\dot{R}/R$, h_{ik} is the chronometrically invariant metric tensor, h is the chronometrically invariant fundamental determinant, ε_{ik} is the viscous stress tensor, $\varepsilon^i_i=0$, $T_{00}=\rho g_{00}$, $c^2T_{ik}=ph_{ik}-\varepsilon_{ik}$, $T_{\mu\nu}$ is the energy-momentum tensor of the medium, ${}^*\nabla_i$ is a symbol of the chronometrically invariant covariant derivative; the Greek-letter indices take on values of 0, 1, 2, and 3, the Latin-letter indices take values only of 1, 2, and 3. The chronometrically invariant differentiation operators are designated by asterisks, the dot designates differentiation with respect to time coordinate t. Then

$$\frac{{}^*\partial}{\partial t}=\frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t},\quad \frac{{}^*\partial}{\partial x^i}=\frac{\partial}{\partial x^i}-\frac{g_{0i}}{cg_{00}}\frac{\partial}{\partial t}.$$

We shall also employ a theory, according to which, in a four-dimensional space-time domain with $g^{00} \neq 0$, system of equations (2)-(4) is equivalent to system of equations (1)-(2), provided that Eqs. (1) are satisfied by virtue of initial conditions (see, for example, the book by Synge [4]).

We shall assume that in the four-dimensional space-time domain under study $g^{00} \neq 0$, that the initial Cauchy hypersurface is specified by the equation $x^0 = 0$, that a regular finite minimum of R is attained on it and the initial conditions on it satisfy signature conditions of Eqs. (1). Quantities $g_{\mu\nu}$ and $T_{\mu\nu}$ will be assumed to be analytic functions of the four coordinates, and the solution of Eqs. (2)-(4) shall be sought in the class of analytic functions.

2. **Case of Viscous Medium (Complete Specification of the Viscosity Law).** Upon specifying holomorphic functions K, η, we impose nine constraints on $T_{\mu\nu}$: $p = K(\rho)$ - equation of state, $\varepsilon_{ik} = 2\eta\Pi_{ik}$ (η is the first viscosity coefficient, $\eta > 0$). $T^i_0 = 0$ is the condition of co-motion of the frame of reference; we also specify $g_{00} = 1$. The unknowns here are functions g_{ik}, ρ, and g_{0i}, which are found from Eqs. (2)-(4). Equations (2) at $g^{00} \neq 0$ consist of equations of the second order in t, with respect to g_{ik}, reduced to normal form; Eq. (3) is the normal equation, of the first order in t, with respect to ρ, Eqs. (4) are equations of the second order in t with respect to g_{0i}, reduced to normal form at $g_{0i} \neq 0$. Thus, Eqs. (2)-(4) comprise a system of 10 equations with respect to g_{ik}, ρ, and g_{0i}. As initial conditions on the hypersurface $x^0 = 0$ we specify the values of 19 quantities, which are holomorphic functions of three variables: $g_{ik}, \dot{g}_{ik}, \rho, g_{0i}, \dot{g}_{0i}$; here they shall be selected in such a manner that: a) the right-hand sides of Eqs. (2)-(4), reduced to normal form, be real and holomorphic functions in the vicinity of the system of values of their arguments in some point $P(x^1_0, x^2_0, x^3_0)$ of the space at $x^0 = 0$; b) four equations (1) would be satisfied at $x^0 = 0$ by virtue of the initial conditions; c) that conditions $^*\dot{R} = 0$, $^*\ddot{R} \geq 0$ of the regular minimum of R be satisfied at $x^0 = 0$. When condition a) is satisfied, according to the Cauchy-Kovalevskaya theorem (see, for example, the book by Petrovskii [5]), in some four-dimensional region containing point $O(0, x_0^1, x_0^2, x_0^3)$, there exists a system of holomorphic functions g_{ik}, ρ, and g_{0i}, satisfying Eqs. (2) and (4) and the initial conditions. When condition b) is satisfied, this system of functions satisfies also Eqs. (1) in some four-dimensional region containing point O. When condition c) is satisfied, the regular minimum of R is attained on hypersurface $x^0 = 0$. The condition $^*\dot{R} = 0$ at $x^0 = 0$ yields

$$-(\dot{g}_{ik})_0(g^{ik})_0 + 2(\dot{g}_{0i})_0(g_{0k})_0(g^{ik})_0 = 0,$$

which represents one constraint on the initial conditions. (Quantity $(g_{\mu\nu})_0$ is the value of $g_{\mu\nu}$ at time $x^0 = 0$). The condition $^*\ddot{R} \geq 0$ at $x^0 = 0$ singles out a domain in the space of initial conditions.

To be able to calculate the number of physically arbitrary functions (i.e., arbitrary functions in a maximally simplified coordinate system of the given frame of reference), one must know which arbitrariness in the selection of the coordinate system is still permissible. The presence of the regular minimum of R on hypersurface $x^0 = 0$, together with the condition that $g_{00} = 1$ fully defines the selection of the time coordinate, so that it becomes possible to perform arbitrary transformations of three spatial coordinates one in terms of the other: $\tilde{x}^i=\tilde{x}^i(x^1, x^2, x^3)$, by means of which three quantities from the initial conditions can be reduced to prespecified values. Thus, the solution of Einstein's equations (1)-(2) obtained contains the maximum number (eleven) of physically arbitrary functions of three variables. The fact that the maximum number of physically arbitrary functions of three variables in solutions of Einstein's equations upon complete specification of the law of viscosity and $\eta \neq 0$ is 11 was shown by Grishchuk [6].

3. **Case of Ideal Medium (η = 0).** We specify, in some space-time domain, 10 quantities: $g_{00}=1$, $p=K(\rho)$, $\varepsilon_{ik}=0$, $T_0^i = 0$. The unknown quantities are g_{ik}, ρ, and g_{0i}. In the case of an ideal medium, Eqs. (3)-(4) are:

$$\dot{\rho} = -3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) \equiv \Psi(x^\alpha), \tag{5}$$

$$\dot{g}_{0i} = \left(\frac{\partial p}{\partial x^i} - \frac{g_{0i}}{c}\dot{p}\right)\frac{1}{c\left(\rho + \frac{p}{c^2}\right)} \equiv \chi_i(x^\alpha). \tag{6}$$

Equations (2) contain the quantities $\frac{\partial \dot{g}_{0i}}{\partial x^\kappa}$ (but do not contain quantity $\ddot{g}_{0i}$); this means that the highest order of derivatives of g_{0i} is the second order. It is hence incorrect to state the Cauchy problem for unknown functions g_{ik}, ρ, and g_{0i} for system of equations (2), (5) and (6). In order for the formulation of the Cauchy problem for the given unknown functions with respect to the set of equations containing Eqs. (2) be proper, it is necessary to have a second-order equation in t with respect to g_{0i}. It can be easily shown that, if functions ρ, g_{0i}, and the right-hand sides Ψ and χ_i of normal equations (5), (6) are holomorphic within some four-dimensional domain, then system of equations (5)-(6) in this domain is equivalent to the system

$$\ddot{\rho} = \dot{\Psi}, \qquad \ddot{g}_{0i} = \dot{\chi}_i, \tag{7}$$

$$(\dot{\rho}_0) = (\Psi)_0, \qquad (\dot{g}_{0i})_0 = (\chi_i)_0. \tag{8}$$

Then Eqs. (2) and (7) comprise a system of ten equations of the second order in t with respect to g_{ik}, ρ, and g_{0i}, reducible to normal form. We specify, as initial conditions on hypersurface $x^0 = 0$, 20 quantities, which are holomorphic functions of three variables: g_{ik}, $\dot{g}_{ik}$, ρ, $\dot{\rho}$, g_{0i}, $\dot{g}_{0i}$, selecting them in such a manner that: the right-hand sides of normalized equations (2) and (7) be real and holomorphic functions in the vicinity of a system of values of their arguments in some point P (x_0^1, x_0^2, x_0^3) of space $x^0 = 0$; there be satisfied conditions b) and c) of Sec. 2; four conditions (8) be satisfied by virtue of initial conditions. Then the solution of system of equations (1)-(2), which exists in some four-dimensional domain, containing point O $(0, x_0^1, x_0^2, x_0^3)$, satisfies initial conditions, and contains a maximum number (eight) of physically arbitrary functions of three variables.

4. **Solution of Einstein's Equations Near the Regular Minimum of R.** We now consider, near the regular minimum of R, the nature of distortion of the co-moving space, the time dependence of volume V and the density of an ideal and of a viscous medium upon complete specification of the law of viscosity. The solution close to (with respect to time) the regular minimum of R, which is attained at $x^0 = 0$, is:

$$g_{ik} = (g_{ik})_0 + (\dot{g}_{ik})_0 t + \frac{1}{2} (\ddot{g}_{ik})_0 t^2 + \ldots,$$

$$g_{0i} = (g_{0i})_0 + (\dot{g}_{0i})_0 t + \frac{1}{2} (\ddot{g}_{0i})_0 t^2 + \ldots,$$

$$\rho = (\rho)_0 + (\dot{\rho})_0 t + \frac{1}{2} (\ddot{\rho})_0 t^2 + \ldots,$$

where the coefficients of powers of t are either Cauchy conditions, or quantities, expressed in terms of Cauchy conditions by means of Eqs. (2)-(4). The solution contains the maximum number of physically arbitrary functions of three variables. At a sufficiently small $|t|$ the value of ρ will be positive if $(\rho)_0$ is specified as positive, and the signature conditions will be satisfied, if they were satisfied at $x^0 = 0$.

The conditions of the regular minimum of R in an orthogonal three-dimensional coordinate system at $x^0 = 0$ become:

$$\frac{(\dot{h}_{11})_0}{(h_{11})_0} + \frac{(\dot{h}_{22})_0}{(h_{22})_0} + \frac{(\dot{h}_{33})_0}{(h_{33})_0} = 0, \tag{9}$$

$$\frac{(\ddot{h}_{11})_0}{(h_{11})_0} + \frac{(\ddot{h}_{22})_0}{(h_{22})_0} + \frac{(\ddot{h}_{33})_0}{(h_{33})_0} \geqslant 0. \tag{10}$$

Satisfaction of Eq. (9) is possible in the following cases: 1) near the regular minimum of R the space is compressed along one coordinate axis and expands along the two others, or conversely (two of the quantities $(\dot{h}_{11})_0$, $(\dot{h}_{22})_0$, $(\dot{h}_{33})_0$ are of the same sign); 2) compression occurs along one axis, expansion takes place along the other, whereas along the third axis the rate of distortion of the space along the third axis at $x^0 = 0$ is equal to zero; 3) the rate of distortion of the space along all the coordinate axes at $x^0 = 0$ is zero. Cases 2 and 3 occur only for specially selected initial conditions. Inequality (10) defines the constraint on acceleration of the distortion of the space at $x^0 = 0$ along different coordinate axes.

The volume of an element of the space changes, near the regular minimum of R, as

$$V = (V)_0 + \frac{1}{2} (\ddot{V})_0 t^2 + \ldots, \quad (\ddot{V})_0 \geqslant 0.$$

It can be easily seen from Eq. (3) that in the case of an ideal medium $(\dot{\rho})_0 = 0, (\ddot{\rho})_0 \leqslant 0$, then density ρ near the regular minimum of R changes as $\rho = (\rho)_0 + (1/2) \cdot (\ddot{\rho})_0 t^2 + \ldots$; this means that the time of density maximum coincides with the time of minimum of the volume. However, in the case of a viscous medium these two times are not the same, and the viscosity changes as $\rho = (\rho)_0 + (\dot{\rho})_0 t + (1/2)(\ddot{\rho})_0 t^2 + \ldots$; here $(\dot{\rho})_0 > 0$.

The scalar Einstein's equation in the chronometrically invariant form at $x^0 = 0$ can be written as [3]

$$3 \left(\frac{{}^*\ddot{R}}{R} \right)_0 + (\Pi_{ik}\Pi^{ik})_0 - (A_{ik}A^{ik})_0 + \left({}^*\nabla_j F^j - \frac{1}{c^2} F_j F^j \right)_0 = -\frac{\varkappa}{2} (\rho c^2 + 3p)_0 + \Lambda c^2.$$

Since ${}^*\ddot{R} \geqslant 0$, the initial value of density $(\rho)_0$ on hypersurface $x^0 = 0$, where the regular minimum of R is attained, and $\Pi_{ik}\Pi^{ik} \geqslant 0$, precisely by virtue of presence of rotation (A_{ik} is the chronometrically invariant tensor of the angular velocity) or negative physical divergence of the gravity-inertial force at $x^0 = 0$, can be specified as positive.

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