A PENROUSE-TYPE DIAGRAM FOR ALGEBRAIC CLASSIFICATION OF ELECTROMAGNETIC FIELDS

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Vol. 36, No. 2, pp. 89-92, 1981

UDC 530.12:531.51

An algebraic classifcation of electromagnetic fields based on investigating characteristic λ matrices for field tensor $F_{\mu\nu}$ is considered. A Petrov-Penrouse type diagram for electromagnetic fields is constructed. An analogy between certain exact solutions for the gravitational and electromagnetic fields is examined.

The algebraic classification of Einstein spaces, established by Petrov [1], has made (and is still making) an important contribution to the development of the theory of gravity. This classification was found to be extremely fruitful for finding new exact solutions of Einstein's equations, and also for working gravitational radiation criteria [2].

An analogous algebraic classification can be also developed for the electromagnetic field, which was frequently mentioned by a number of investigators [3-5]. However, such a classification of the electromagnetic field, based on investigating characteristic λ matrices has not yet been published, although an identical Petrov classification for the gravitational field is currently in general use. For example, Bilyalov [5] considers in passing a classification of the electromagnetic field, but it is poorer than the classification performed with λ matrices. In particular, it does not at all contain type D electromagnetic fields (see below).

The purpose of the present paper consists in extending the Petrov classification to the electromagnetic field. We construct a diagram, analogous to the Penrouse diagram [6] for the gravitational field. This approach is also suitable for algebraic classification of gravity-inertial waves. This analogy with electromagnetic waves supports the Likhnerovich criterion, according to which gravitational waves are described by Petrov's N subtype metrics.

We now investigate the algebraic structure of the electromagnetic field tensor $F_{\mu\nu}$ on the basis of an investigation of the λ matrix: $|F_{\mu\nu}-\lambda g_{\mu\nu}|$. The type of the field will be determined by the characteristic of the λ matrix, reduced to canonical form· it is well preserved in the domain where this characteristic does not change. Characteristic equation $\|F_{\mu\nu}-\lambda g_{\mu\nu}\|=0$ will be written in the form $\lambda^4+I_1\lambda^2-I_2{}^2=0$, where

$$I_1=H^2-E^2\equiv F_{\mu\nu}F^{\mu\nu};\ I_2=(\mathbf{E},\mathbf{H})\equiv\varepsilon^{\alpha\beta}_{\cdot\cdot\mu\nu}F^{\mu\nu}F_{\alpha\beta}.$$

Let point P have corresponding to it a λ matrix with a system of elementary divisors:

$$\begin{cases} (\lambda-\lambda_1)^{m_1},\ (\lambda-\lambda_2)^{m_1'},\ (\lambda-\lambda_3)^{m_1''}\ldots \\ (\lambda-\lambda_1)^{m_2},\ (\lambda-\lambda_2)^{m_2'}\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Then this λ matrix has corresponding to it the characteristic $[(m_1, m_2 \ldots), (m_1', m_2' \ldots) \ldots]$, which defines the type of the space in that domain A, where it does not change. Invariants λ_i, which are the bases of elementary divisors, are at the same time the roots[1] of the characteristic equation: $\|F_{\mu\nu}-\lambda g_{\mu\nu}\|=0$. However, in domain A it is possible to have coincidence of elementary divisors on certain manifolds of bases $\lambda_i(x)$. These manifolds will be described by equations such as: $\lambda_i(x)-\lambda_j(x)=0$. The elementary divisors will be in the form $(\lambda-\lambda_1)^p, (\lambda-\lambda_2)^q \ldots$, the characteristic will be [p,q,...], and on the hypersurface under consideration [p(q)...] (i.e., in domain A in general $\lambda_1 \neq \lambda_2$, but in some subdomain they are equal, and hence p and q correspond to the same basis divisor). By direct reduction of the λ to canonical form we obtain, in the case of $I_1 \neq 0$, $I_2 \neq 0$:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda^4+I_1\lambda^2-I_2^2 \end{pmatrix}. \tag{1.1}$$

This form of the λ matrix is also retained for the case of $I_1 = 0$, $I_2 \neq 0$. In the case of $I_1 \neq 0$, $I_2 = 0$ we find the following canonical form of the λ matrix, given in Eq. (1.2). This form is retained for the case of $I_1 = 0$, $I_2 = 0$; it is given by Eq. (1.3).

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda(\lambda^2+I_1) \end{pmatrix} \tag{1.2}$$

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda^3 \end{pmatrix} \tag{1.3}$$

It is important to note that for the electromagnetic field it is impossible to have a type defined by characteristic [(2,2)], i.e., by a matrix such as:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & \lambda^2 \end{pmatrix},$$

since it necessarily follows from direct reduction of the λ matrix to canonical form in the case of $I_1 = 0$, $I_2 = 0$, that for such a matrix $\mathbf{H}\|\mathbf{E}$, but, since $I_2=0 \Rightarrow \mathbf{H}\perp\mathbf{E}$ and given $I_1=0 \Rightarrow H=E=0$, we find that such a situation can arise only for the trivial case.

The Penrouse diagram for classification of the gravitational field using λ matrices with respect to the curvature tensor or Weyl's tensor (Fig. 1) (we write only the first half of the characteristic, dropping the complex-conjugate part), is quite familiar. An analogous diagram was constructed for the electromagnetic field (Fig. 2).

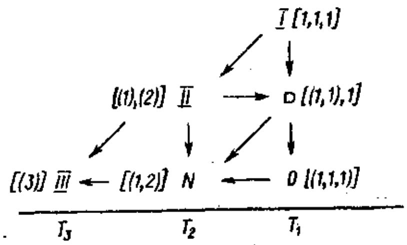

Fig. 1

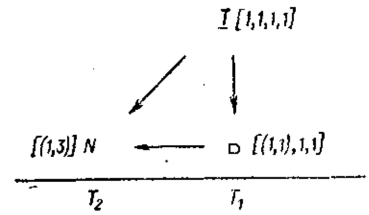

Fig. 2

Diagram 2 allows a fresh insight into certain types of the gravitational field. For a type N[(1,3)] electromagnetic field $I_1 = I_2 = 0$, i.e., $E = H$, $\mathbf{E} \perp \mathbf{H}$; which means that type N contains electromagnetic waves. (In general type N is not limited to electromagnetic waves.) According to its characteristic, it has corresponding to it a type N gravitational field. It is hence natural to assume that gravity waves should be described by metrics, pertaining to type N gravitational field. Also, an electromagnetic field of type D corresponds to the case of $I_2 = 0$; in particular, the Coulomb solution for H = 0 is included here. The Schwarzschild metric is also contained in the set of type D[(1,1)1] spaces and corresponds to the Coulomb field. Note that it does not exhaust the entire type D.

It is interesting that, of the entire set of possible canonical types of λ matrices only three "perservere" for the electromagnetic field. The question hence arises as to why other types are not realized. This is apparently related to the properties of electromagnetic field tensor $F_{\mu\nu}$. In particular, instead of the two types: N[(1,3)] and III[1,3] there remains only the former, whereas type III cannot be realized due to antisymmetry of the electromagnetic field tensor, and in particular, due to equality to zero of the coefficient of λ^3 in the characteristic equation: we obtain $\lambda^4 = 0$ at $I_1 = I_2 = 0$, i.e., all the four roots are identical. An analogous situation is also observed for certain other types of the electromagnetic field, which are formally expected.

The author is deeply indebted to his scientific advisor, Doctor of Physical and Mathematical Sciences, Yu. S. Vladimirov, for statement of the problem, discussion of results, and constant attention to the work.

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1 September 1980

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