VON NEUMANN ALGEBRAS FOR ARBITRARY-SPIN FIELDS

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A modification of Khoruzhii's method is formulated for a system of Whiteman fields of arbitrary spin, which enables one to use the modular Tomita-Takesaki theory to put the Whiteman fields into correspondence with local von Neumann field algebras without imposing additional conditions on the fields. This provides a network of von Neumann algebras for fields of any spin that satisfies all the axioms in the algebraic approach, including the axiom of locality in the generalized form of locality with torsion.

One of the main problems in rigorous research on the structure of quantum-field models it to obtain a system of local von Neumann field algebras $\mathcal{F}(O)$ for a given model. To solve this problem requires sets of unbounded operators $\Pi(\hat{f})$ that are put into correspondence with the set of von Neumann algebras $\mathcal{F}(O)$ while retaining the axioms of relativistic quantum theory. Usually, in order to ensure the existence and necessary properties of $\mathcal{F}(O)$ for a given set of fields $\Pi(\hat{f})$, various conditions of technical character are imposed on these fields. Khoruzhii has developed a new method of introducing the $\mathcal{F}(O)$ algebras that does not require any additional conditions and which is based on using the best tool in the current theory of operator algebras: the modular Tomita-Takesaki theory. A paper will be published in which this method is developed for the case of a system of neutral fields. In the present note, we consider the case of a general system of Whiteman fields with any spin.

1. In a preceding paper [1], we constructed a Borhers algebra $\Omega_0{}^*$. The Whiteman functional W given in $\Omega_0{}^*$ canonically defined the representation Π of the algebras $\Omega_0{}^*$ by means of unbounded operators for Whiteman fields of any spin. Now our task is to construct a network of field von Neumann algebras $\mathcal{F}(O)$ for the previous system of unbounded field operators $\Pi(\hat{f})$. To apply Khoruzhii's method, we consider a network of Op* algebras $\mathcal{P}(O)$ generated by the fields

$$\Pi(\hat{\mathbf{f}}),\ \hat{\mathbf{f}}\in\Omega_0^*(O)=\bigoplus_{n=0}^{\infty}(S(O)^{\oplus r})^{\otimes n}\subset\Omega_0^*.$$

As locality with torsion applies for this net,

$$\forall O_1\sim O_2\quad \mathcal{P}(O_1)\subset\mathcal{P}(O_2)^{z\prime},\ \mathcal{P}(O_2)^z=Z\mathcal{P}(O_2)Z^{-1},\tag{1}$$

where according to [1] the torsion operator Z takes the form

$$Z\xi(\hat{\mathbf{f}})=\xi((1+i)^{-1}(\hat{\mathbf{f}}+i\hat{\mathbf{f}}^{(0,-1)})),$$

and it is readily found (see [2], note to theorem 4.3) that the Ray-Schlider theorem applies for $\mathscr{P}(O)$, i.e., ξ_0 is a bicyclic vector for $\mathscr{P}(O)$. Also, we introduce the set $\tilde{\mathscr{U}}(O)\equiv\mathscr{P}(O)\xi_0$ and assign it the structure of an *-algebra by introducing the involution

$$\forall\xi=\Pi_\xi\xi_0\in\tilde{\mathscr{U}}(O), \qquad \xi\to\xi^*=\Pi^*{}_\xi\xi_0$$

and the multiplication

$$\lambda\xi=\Pi_\lambda\Pi_\xi\xi_0.$$

We show that the involutions $\tilde{S}$ in $\tilde{\mathscr{U}}(O)$ allows closure (in view of the replacement of locality by locality with torsion, the corresponding proof differs from that for the scalar case). It is necessary to check that $\xi = 0$ if the sequence $\{\xi_n\}_{n=1}^{\infty}$ from $\mathscr{D}[S]$ strongly converges to 0 in $\mathscr{H}$ and $\tilde{S}\xi_n$ strongly converges to $\xi\in\mathscr{H}$. For any $\lambda\in\mathscr{P}(O')^z\xi_0$, we use the locality with torsion and (1) to get

$$(\lambda, \tilde{S}\xi_n)=(\Pi\hat{f}_\lambda)^z\xi_0, \ \Pi(\hat{f}_n)^*\xi_0)=(\Pi(\hat{f}_n)\xi_0, \ \Pi(\hat{f}_\lambda)^*\xi_0),$$

which means that $\xi_n\to 0$ implies $(\lambda, \tilde{S}\xi_n)=(\lambda, \xi)=0$ for all $\lambda\in\mathscr{P}(O')^z\xi_0=Z\mathscr{P}(O')Z^{-1}\xi_0$. However, the Ray-Schlider theorem for $\mathscr{P}(O')$ and the unitarity of Z imply compactness of set $\mathscr{P}(O')^z\xi_0$ in $\mathscr{H}$. We therefore have the desired result $\xi = 0$. As a consequence, as in the scalar case, the set $\tilde{\mathscr{U}}(O)$ is a generalized quasi-Hilbert algebra.

2. In order to define the algebra $\mathscr{F}(O)$ by Khoruzhii's method, we distinguish in the algebras $\mathscr{P}(O)$ systems of symmetrical and positive generators. The general properties of σ algebra representations imply that all the Hermitian elements $\hat{f}=\hat{f}^*$ from algebra $\Omega_0{}^*$ are symmetrical operators $\Pi(\hat{f})$ in $\mathscr{H}$ (in the general case $\Pi(\hat{f})^*\subset\Pi^*(\hat{f})$). while the Hermitian elements of the form $\hat{g}=\hat{f}^*\hat{f}, \hat{f}\in\Omega_0^*$ (which form a convex cone in $\Omega_0{}^*$) are positive operators (in fact, for all $\xi\in\mathscr{D}$, $((\xi, \Pi(\hat{f}^*\hat{f})\xi)=\|\Pi(\hat{f})\xi\|\geqslant|\geqslant 0)$. There is a difference from the scalar case, however, in that involution in $\Omega_0{}^*$, and in particular also in $\Omega_0{}^{(1)*}$, does not now coincide with the simple complex conjugation of functions, and consequently the field operators $\Pi(\hat{f})$, $\hat{f}\in\Omega_0{}^{(1)*}$, taken on the real basic functions $(\hat{f}=(0, f_i, 0, \ldots);\ f_i(x)\in S_R(R^4),\ i=1, 2, \ldots, r)$ are no longer symmetrical.

We transfer to the symbols of the modular Tomita-Takesaki theory, and introduce the new symbols $\pi(\lambda)\equiv\Pi(\hat{f}_\lambda), \lambda\in\tilde{\mathscr{U}}(O)$, for the field operators from $\mathscr{P}(O)$, and we now consider these operators as ones that realize the representation π of the generalized quasi-Hilbert algebra $\tilde{\mathscr{U}}(O)$. This new definition is shown to be correct by the fact that the vector ξ_0 is bicyclic for $\mathscr{P}(O)$. The Hermitian elements $\hat{f}=\hat{f}^*\in\Omega_0^{*(0)}$ correspond to the hermitian elements $\lambda=\lambda^*\in\tilde{\mathscr{U}}(O)$ and the symmetrical operators $\pi(\lambda)\in\mathscr{P}(O)$ with region $\mathscr{D}$. The following formula defines the von Neumann field algebra for a bounded open region O in Minkowski M:

$$\mathscr{F}(O)=\{\exp[it\,\tilde{\pi}(\lambda^*\lambda)]\,|\,t\in R^1, \ \lambda\in\tilde{\mathscr{U}}(O)\}'', \qquad (2)$$

where "~" denotes the Friedrichs extension of the positive operator:

$$\pi(\lambda^{\#}\lambda)=(\Pi^*(\hat{f}_\lambda)\,\Pi(\hat{f}_\lambda))|_{\tilde{\mathscr{U}}(O)}.$$

3. We establish the properties of the net $\{\mathscr{F}(O)\}$ by modifying the method proposed for the scalar case on the basis of the generalized form for the locality axiom and the presence in our scheme of a finite-dimensional representation V(A) of the group SL(2, C), which is trivial in the scalar case (V(A) = E). First

of all, we introduce involution and multiplication in an obvious fashion in the set $\tilde{\mathcal{U}}(O)^z \equiv \mathcal{P}(O)^z \xi_0 \subset \mathcal{D}$, which readily shows, as in the case of $\tilde{\mathcal{U}}(O)$, that involution on $\tilde{\mathcal{U}}(O)^z$ is closable, and that $\tilde{\mathcal{U}}(O)^z$ is a generalized quasi-Hilbert algebra. Hence we get an equivalent representation of the algebra $\mathcal{F}(O)$ by means of a positive cone $P(O')^Z$ in algebra $\tilde{\mathcal{U}}(O')^z$:

$$P(O')^z = \{x \in \tilde{\mathcal{U}}(O')^z \mid x = \lambda^* \lambda, \quad \lambda \in \tilde{\mathcal{U}}(O')^z\}.$$

For each $x \in P(O')^z$ we introduce a positive quadratic form q_x with region $\mathcal{D}$:

$$q_x(\xi, \eta) = (\xi, \pi(x)\eta); \ \xi, \eta \in \mathcal{D}$$

(here this is already the π representation of the algebra $\tilde{\mathcal{U}}(O')^z$ and $\pi(x)$ is a positive operator). The form q_x defines the Hilbert space

$$\mathcal{H}_x \equiv \overline{\mathcal{D}}^{\|\cdot\|_x}, \ (\xi, \eta)_x = (\xi, \eta) + q_x(\xi, \eta) = (\xi, (E + \pi(x))\eta).$$

For the operators $\pi(\lambda^* \lambda)$, $\lambda \in \tilde{\mathcal{U}}(O)$, we define their contraction $\pi_x(\lambda^* \lambda)$ on $\mathcal{H}_x$ (in view of the closability of form q_x, $\mathcal{H}_x \subset \mathcal{H}$) and construct the Friedrichs expansions $\tilde{\pi}_x(\lambda^* \lambda)$ of these contractions, which are positive operators in $\mathcal{H}_x$. Then the resulting positive self-conjugate operator in $\mathcal{H}_x$ is put into correspondence with the operators $j[\tilde{\pi}_x(\lambda^* \lambda)]$ in $\mathcal{H} \supset \mathcal{H}_x$ having the same region of action and the same law for it. It can be shown that $j[\tilde{\pi}_x(\lambda^* \lambda)]$ is an essentially self-conjugate operator in $\mathcal{H}$, and its closure coincides with the operator $\tilde{\pi}(\lambda^* \lambda)$. We put

$$\exp[it\tilde{\pi}(\lambda^* \lambda)] \equiv u_t^\lambda, \ \exp[it[\tilde{\pi}_x(\lambda^* \lambda)]] \equiv u_t^{\lambda, x},$$

to get the desired representation for the algebra $\mathcal{F}(O)$:

$$\begin{aligned} \mathcal{F}(O) &= \{u_t^\lambda \mid t \in R^1, \ \lambda \in \tilde{\mathcal{U}}(O)\}'' = \\ &= \{u_t^{\lambda, x} \mid t \in R^1, \ \lambda \in \tilde{\mathcal{U}}(O), \ x \in P(O')^z\}. \end{aligned}$$

This representation, as in the scalar case, gives us a proof for the first of the basic properties of $\{\mathcal{F}(O)\}$: the vacuum vector is a bicyclic vector for any algebra $\mathcal{F}(O)$.

We can modify the proof of the locality property proposed for the scalar case by replacing $P(O)$ by $P(O)^Z$ and $\tilde{\mathcal{U}}(O)$ by $\tilde{\mathcal{U}}(O)^z$, which gives us further that there is locality with torsion for $\{\mathcal{F}(O)\}$: $\mathcal{F}(O') \subset \mathcal{F}(O)^{z'}$, where the torsion operator Z can be taken as for the net .

As

$$\tilde{\pi}(\lambda^* \lambda) = \Pi(\hat{f}^* \hat{f})|_{\tilde{\mathcal{U}}(O)},$$

we get the property of relativistic invariance by Khoruzhii's method.

The proof of the other axioms in the algebraic approach also does not require any changes, and we obtain the desired result: the net of von Neumann field algebras $\mathcal{F}(O)$ put via (2) into correspondence with a system of Whiteman fields of any spin satisfies all the axioms in the Haag-Araki algebraic approach (with the locality axiom in generalized form). Here the vacuum vector in the Whiteman theory is bicyclic for all algebras $\mathcal{F}(O)$, $O' \neq \Phi$.

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