THE HAMILTONIAN FORMS OF SOME INTEGRATION METHODS

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Three integration methods are considered in Hamiltonian form: the rapidly rotating phase method, generalization of the Newton-Kantorovich method for arbitrary systems, and Fock's fifth-parameter method, which opens up a new approach to integrating Maxwell's equations.

Approximate methods are mainly used to solve problems in theoretical physics, because a few exact solutions often have forms unsuitable for numerical calculations. On the other hand, there are no general methods of solving equations containing nonlinearities, variable coefficients, or arbitrary boundary conditions. In that case, perturbation theory is a basic tool for a wide range of problems in physics. Examination of expansion structure as a whole makes it necessary to construct higher approximations. This task can be handled in general form for Hamiltonian systems on the basis of canonical perturbation theory [1].

Here we consider three integration methods represented in Hamiltonian form: Fock's fifth-parameter method [2-4], the rapidly-rotating phase method [5,6], and the Newton-Kantorovich method [7].

1. CANONICAL PERTURBATION THEORY

Let R^{2s} be a phase space with coordinates $x^k = z^k$ and momenta $p_k = z^{k+s}$, in which we are given an oblique-scalar metric and the Hamiltonian $H = H(z, \tau)$. The fundamental Poisson brackets PB are $[z_\mu, z_\nu] = \Omega_{\mu\nu}$ [8]. Any dynamic quantity $F(z, \tau)$ satisfies

$$\frac{dF}{d\tau} = \frac{\partial F}{\partial \tau} + [F, H]. \tag{1}$$

We represent H in the form $H = H_0 + \Delta H$, where H_0 is a Hamiltonian allowing of exact integration, $z = z(z', \tau)$. If the fundamental PB calculated with respect to the variable z' are conserved, the substitution $z \to z'$ is a canonical transformation CT. The variables z' satisfy (1) with the Hamiltonian $H'(z', \tau) = \Delta H(z(z', \tau), \tau)$. According to [1], the solution to (1) can be put as the series

$$F(\tau) = \sum_n \int_{\tau_0}^{\tau} d\tau_1 \int_{\tau_0}^{\tau_1} d\tau_2 \dots \int_{\tau_0}^{\tau_{n-1}} d\tau_n [\dots [F'(z'_0, \tau), H'(z'_0, \tau_1)], \\ H'[(z'_0, \tau_2)] \dots] H'(z'_0, \tau_n)], \tag{2}$$

where $F'(z', \tau) = F(z(z', \tau), \tau)$.

The convergence of (2) can be evaluated by the majorant method on the basis of Cauchy's theorem [9] or the compressed mapping principle [10].

2. THE FIFTH-PARAMETER METHOD

We consider an equation in first-order partial derivatives for the phase of a normal wave propagating in a nonstationary anisotropic inhomogeneous medium with space-time dispersion [11],

$$H(p_\mu, x^\mu) \equiv (\partial_n \psi)^2 - \varepsilon(p_\mu, x^\mu)(\partial_0 \psi)^2 = 0, \tag{3}$$

where the wave 4-vector is $p_\mu = -\partial\psi/\partial x^\mu$. We introduce the parameter τ and replace (3) by an equation in five variables:

$$\partial_\tau \psi + H(p_\mu, x^\mu) = 0. \tag{4}$$

The complete integral of (4) is $\psi = \psi(x^\mu, p'_\mu, \tau) + c$, where p'_μ is a constant 4-vector and this defines the ray path in parametric form as

$$p_\mu = -\frac{\partial\psi}{\partial x^\mu}, \quad x'^\mu = -\frac{\partial\psi}{\partial p_\mu}. \tag{5}$$

The complete integral should satisfy the condition $\partial_\tau \psi = 0$.

The characteristic system corresponding to (5) has the canonical form

$$\frac{dx^\mu}{d\tau} = -\frac{\partial H}{\partial p_\mu}, \quad \frac{dp_\mu}{d\tau} = \frac{\partial H}{\partial x^\mu}. \tag{6}$$

In fact, on substituting the complete integral into (4) we get an identity, which we differentiate with respect to x^μ:

$$\frac{\partial^2\psi}{\partial\tau\partial x^\mu} + \frac{\partial H}{\partial x^\mu} + \frac{\partial H}{\partial p_\nu}\frac{\partial p_\nu}{\partial x^\mu} = 0. \tag{7}$$

If now the tangent to the curve of (5) is taken in accordance with the first equation in (6), then (7) becomes the second equation of (6). Expression (2) enables us to derive the law of motion for the ray and the phase and intensity along the ray path.

Example. Deflection of a light beam by the gravitational field. In that case, the Hamiltonian is [12]

$$H(x, p) = -\frac{1}{2} g^{\mu\nu}(x) p_\mu p_\nu.$$

We represent H in the form $H = H_0 + \Delta H$, where

$$H_0 = -\frac{1}{2}\eta^{\mu\nu} p_\mu p_\nu, \quad \Delta H = -\frac{1}{2} h^{\mu\nu} p_\mu p_\nu,$$

and $\eta^{\mu\nu}$ is the planar-space metrical tensor. The expression for the metric $h^{\mu\nu} = g^{\mu\nu} - \eta^{\mu\nu}$ of a centrally symmetrical field is as follows in the linear approximation:

$$h_{00} = -r_g/r, \quad h_{mn} = (-r_g/r)\delta_{mn}, \quad r = \sqrt{x^2 + y^2 + z^2}.$$

We perform the CT $x = x' + p'\tau$, $p = p'$. Then the evolution of the primed variables is determined by ΔH. We put $F = p_\mu$ in (2) to get

$$p_\mu = p'_\mu(0) - \int_{\tau_0}^{\tau} d\tau_1 \frac{\partial h^{\alpha\beta}}{\partial x'^\mu} p'_\alpha p'_\beta + \dots .$$

We assume that the initial conditions take the form $x^\mu = (-\infty, -\infty, b, 0)$, $p^\mu = (p_0, p_0, 0, 0)$. We integrate to get the momentum increment

$$\Delta p_y = -b p_0^2 r_g \int_{-\infty}^{\infty} \frac{d\tau}{(b^2 + p_0^2\tau^2)^{3/2}} = -\frac{2r_g}{b} p_0.$$

Consequently, the deflection angle is proportional to $2r_g/b$ [12,13].

The fifth-parameter method in the theory of electromagnetic-wave propagation. We write Maxwell's equations for an inhomogeneous anisotropic or gyrotropic medium [14]:

$$(\mathrm{rot\,rot\,}\mathbf{E})_\alpha + \partial_0{}^2 \hat{\varepsilon}_{\alpha\beta} E_\beta = 0, \tag{8}$$

where $\hat{\varepsilon}_{\alpha\beta}$ is an operator that allows for the delay in the polarization of matter relative to the electromagnetic wave. The solution to this equation can be used with the integral representation [2]

$$E_\alpha(t, \mathbf{x}) = \frac{1}{2\pi} \int u_\alpha(\omega, \mathbf{x}, \tau) e^{-i\omega t} d\omega d\tau \tag{9}$$

to relate it to the 3-vector u_α, which satisfies

$$i\partial_\tau u_\alpha + \partial_\beta(\partial_\beta u_\alpha - \partial_\alpha u_\beta) + \omega^2 \varepsilon_{\alpha\beta}(\omega, \mathbf{x}) u_\beta = 0. \tag{10}$$

Then $\partial_\tau u_\alpha = 0$ at the ends of the contour of integration [2].

System (10) is the solution to the variational problem for the functional

$$I = \int \left[\frac{i}{2} (\partial_\tau u^*_\alpha u_\alpha - u^*_\alpha \partial_\tau u_\alpha) - \omega^2 \varepsilon_{\alpha\beta}(\omega, \mathbf{x}) u^*_\alpha u_\beta + |\mathrm{rot\,}\mathbf{u}|^2 \right] d^3x d\tau. \tag{11}$$

We assume that $\varepsilon_{\alpha\beta} = \delta_{\alpha\beta} + \Delta_{\alpha\beta}(\omega, \mathbf{x})$. If $\Delta_{\alpha\beta} = 0$, the solution of (9) takes the form

$$u_\alpha(\omega, \mathbf{x}, \tau) = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=1,2} \int d^3k e^{(\lambda)}_\alpha(\mathbf{k}) a_\lambda(\mathbf{k}) e^{ikx + i(\omega^2 - k^2)\tau}. \tag{12}$$

Here $e_\mu^{(\lambda)}(\mathbf{k})$ are unit vectors in momentum space orthogonal to the wave vector and one to another, where

$$e^{(\lambda)}_\mu e^{(\lambda)}_\nu = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2_\circ}.$$

We seek the solution of (10) in the form of (12) on the assumption that $a_\lambda = a_\lambda(\mathbf{k}, \tau)$. We substitute (12) into (11) and integrate over the spatial coordinates to get the new functional

$$I = \int L d\tau, \quad L = \frac{i}{2} \int d^3k (\dot{a}^*_\lambda a_\lambda - a^*_\lambda \dot{a}_\lambda) + H(a, a^*, \tau),$$
$$H = -\omega^2 \int \Delta_{\alpha\beta}(\omega, \mathbf{x}) u^*_\alpha(\omega, \mathbf{x}, \tau) u_\beta(\omega, \mathbf{x}, \tau) d^3(x). \tag{13}$$

If we introduce a phase space with coordinates $q = a_\lambda$ and momenta $p = ia_\lambda^*$, the Lagrange-Euler equations coincide with the canonical Hamilton ones:

$$\dot{a}_\lambda = [a_\lambda, H], \qquad \dot{a}_\lambda^* = [a_\lambda^*, H].$$

The fundamental PB is $[a^*_{\lambda'}(\mathbf{k}', \tau), a_\lambda(\mathbf{k}, \tau)] = i\delta_{\lambda\lambda'}\delta^{(3)}(k - k')$. Then the solution to (10) is defined by (2), in which we should put $F = u_\alpha(\mathbf{x}, \tau)$:

$$u_\alpha(\mathbf{x}, \tau) = \bar{u}_\alpha(\mathbf{x}, \tau) + \int_{-\infty}^{\tau} d\tau_1 [\bar{u}_\alpha(\mathbf{x}, \tau), \bar{H}(\tau_1)] + \ldots = \\ = \bar{u}_\alpha(\mathbf{x}, \tau) + \omega^2 \int G_{\alpha\beta}(\mathbf{x}, \mathbf{x}')\Delta_{\beta\gamma}(\omega, \mathbf{x}')\bar{u}_\gamma(\mathbf{x}', \tau)\, d^3x'. \qquad (14)$$

The overbar denotes that the phase coordinates should be taken with constant values of $a_\lambda(\mathbf{k}, \tau \to -\infty)$, while $G_{\alpha\beta}$ is Green's function for the equation

$$\partial_\alpha(\partial_\alpha G_{\mu\nu} - \partial_\mu G_{\alpha\nu}) + \omega^2 G_{\mu\nu} = -\delta_{\mu\nu}\delta^{(3)}(\mathbf{x} - \mathbf{x}'),$$

$$G_{\mu\nu}(\mathbf{x}, \mathbf{x}') = \frac{1}{(2\pi)^3}\int d^3k \, \frac{e_\mu^{(\lambda)}(\mathbf{k})\, e_\nu^{(\lambda)}(\mathbf{k})}{\mathbf{k}^2 - k_0 - i0}\, e^{ik(\mathbf{x} - \mathbf{x}')}.$$

We substitute (14) into (9) to get the solution in the form of a Born series.

Describing the field as a mechanical system with a continuous number of degrees of freedom opens up considerable scope for using methods from Hamiltonian dynamics in constructing approximate solutions [15,16]. We perform, for example, a CT to the cordinates ϕ_λ and momenta J_λ by means of the substitution

$$a_\lambda = \sqrt{J_\lambda}\, e^{-i\varphi_\lambda}, \quad a_\lambda^* = \sqrt{J_\lambda}\, e^{i\varphi_\lambda}.$$

We substitute $F = \phi_\lambda$ into (2) to get an expression for the phase corresponding to the smooth-perturbation method [17]. We make two comments. If we perform the Fourier transformation of (8) with respect to the coordinates t, x, and y, we get a Hamiltonian system in which the coordinate z acts as the evolutionary variable. However, the representation of (9) and (12) is very convenient using the Hamiltonian of (13) to examine the propagation and linear interaction of a superposition of noncollinear quasi-planar waves by the parabolic-equation method [17,18].

3. THE FAST-PHASE METHOD

We consider the equation system generated by the Hamiltonian

$$H = \frac{1}{2} p_n p_n + \frac{1}{2}\omega_n^2 x_n x_n + \Pi(x, p, \tau).$$

We perform the CT $x, p \to \phi, J$ on the action-angle variables:

$$x_n = \sqrt{2J_n/\omega_n}\cos\varphi_n, \quad p_n = -\sqrt{2J_n\omega_n}\sin\varphi_n,$$

to get the new Hamiltonian

$$H' = \omega_n J_n + \Pi'(\varphi, J, \tau)$$

and a conical equation system in standard form.

In the fast-phase method it is assumed that the initial conditions correspond to rotational modes of motion (the kinetic energy substantially exceeds the potential). This feature makes it similar to the parabolic-equation method [17,18].

As an example, we consider the equation for the mathematical pendulum $\ddot{x}+\sin x=0$ with initial conditions $x(0)=0$, $\dot{x}(0)=p_0\gg 1$. For this reason, we represent the Hamiltonian $H=(1/2)p^2-\cos x$ as $H = H_0 + \Delta H$, where $H_0 = (1/2)p^2$. The solution to the equations generated by $H_0: x'+p'\tau$, $p=p'$ is a CT for the primed variables. The new Hamiltonian is $H'=-\cos(x'+p'\tau)$. We put F = x in (2) to get the solution in the form

$$x = x_0' + p_0'\tau - \frac{\tau}{p_0'}\cos x_0' + \frac{1}{p_0^2}[\sin(x_0' + p_0'\tau) - \sin x_0'] + \ldots .$$

On using the initial conditions we get

$$x = p_0\tau - \frac{\tau}{p_0} + \frac{1}{p_0^2}\sin p_0\tau + \ldots . \qquad (15)$$

This expression was derived on p. 52 of [6] on the assumption that $p_0 \gg 1$ as a result of extensive calculations in the second approximation by a special method. We note that in contrast to the established viewpoint, the solution of (15) obtained by the fast-phase method describes not only rotation but also oscillation near the equilibrium position x = 0. In fact, because of the maximal analyticity of the force represented by sin x, the series of (15) converges in the region $-\infty < \tau < \infty$ for any p_0. We expand (15) as a series in $p_0 \ll 1$ to get

$$x = p_0\tau - p_0\frac{\tau^3}{3!} + \ldots = p_0 \sin\tau.$$

4. HAMILTONIAN FORM OF THE NEWTON-KANTOROWICZ METHOD

We find the solution to the Cauchy problem for the canonical system. We represent the Hamiltonian as $H = H_0 + \Delta H$ and perform the CT $z = u(z', \tau)$ generated by H_0. In this approximation, the solution to the Cauchy problem is $z_\mu^{(0)} = u_\mu(c, \tau)$, where $c_\mu = z'_\mu$. The coordinates z'_μ satisfy (1) with the Hamiltonian $H'(z', \tau)=\Delta H(z(z', \tau)^\mu, \tau)$. We now expand H' as a Taylor series up to terms of the third order in $(z' - c)_\mu$:

$$H'(z', \tau) = H'(c, \tau) + (z' - c)_\mu \frac{\partial H'}{\partial z_\mu'} + \frac{1}{2}(z' - c)_\alpha (z' - c)_\beta \frac{\partial^2 H'}{\partial z_\alpha' \partial z_\beta'} + \ldots . \qquad (16)$$

Equations corresponding to (16) represent a linear system with variable coefficients and the initial condition $z_\mu'(\tau_0)=c_\mu$. The solution is $z_\mu'=v_\mu^{(1)}(c, \tau)$. Consequently, the approximate solution to the Cauchy problem is $z_\mu^{(1)}=u_\mu(v^{(1)}(c, \tau), \tau)$. We now find the subsequent approximation. For this purpose, we expand the initial Hamiltonian H' at the point $z'=v^{(1)}$ and repeat the above procedure to get the solution $z_\mu'=v_\mu^{(2)}(c, \tau)$. We apply this technique n times to find the approximate solution to the Cauchy problem $z_\mu^{(n)}=u_\mu(v^{(n)}(c, \tau), \tau)$. If we restrict ourselves to the linear approximation in (16), $v^{(n)}$ is defined by the following recurrent relation $(v^{(0)}=c)$:

$$v_\mu^{(n)} = v_\mu^{(n-1)} + \int_{\tau_0}^{\tau} \Omega_{\mu\nu} \frac{\partial H'(z', \tau_1)}{\partial z_\nu'}\bigg|_{z'=v^{(n-1)}(c,\tau_1)} d\tau_1. \qquad (17)$$

A substantial advantage of this integration method is the accelerated convergence typical of the Newtonian tangent method [7].

As an example, we consider the Cauchy problem x(0) = 0 for the equation $\dot{x}=x^2+\tau^2$ [7]. The Hamiltonian is $H=p(x^2+\tau^2)$. We put $H_0 = p\tau^2$ to get the CT

$$x = x' + \tau^3/3, \qquad p = p'$$

and the new Hamiltonian $H' = p'(x' + \tau^3/3)^2$. In the zero-th approximation ($x' = c = 0$), the solution to the Cauchy problem is $x^{(0)} = \tau^3/3$. We use (17) to get subsequently in the linear approximation

$$v^{(1)} = \int_0^{\tau} [x', H'(\tau_1)]_{x'=0} d\tau_1 = \frac{\tau^7}{7\cdot 9}, \quad x^{(1)} = \frac{\tau^3}{3} + \frac{\tau^7}{7\cdot 9},$$
$$v^{(2)} = \int_0^{\tau} [x', H'(\tau_1)]_{x'=v^{(1)}} d\tau_1 = \frac{\tau^7}{7\cdot 9} + \frac{2\cdot\tau^{11}}{3\cdot 7\cdot 9\cdot 11} + \frac{\tau^{15}}{(7\cdot 9)^2\cdot 15}, \quad x^{(2)} = \frac{\tau^3}{3} + v^{(2)}. \tag{18}$$

We note that (18) coincides with the solution obtained by Chaplygin's method (p. 25 of [7]). In the quadratic approximation of (16), the solution to the Cauchy problem

$$x^{(1)} = \frac{\tau^3}{3} + \int_0^{\tau} \left(\frac{\tau_1^3}{3}\right)^2 \exp\left[\int_{\tau_1}^{\tau} \frac{2}{3} u^3 du\right] d\tau_1$$

coincides with that found on p. 28 of [7] by the Newton-Kantorowicz method.

In conclusion, we note that the CT method opens up wide scope for obtaining approximate solutions to systems of equations in ordinary and partial derivatives.

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