FREQUENCY-SELECTIVE ACTION ON NONLINEAR WAVES IN AN ACOUSTIC CAVITY

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A mathematical technique has been developed for analyzing the excitation and interaction of nonlinear waves in an acoustic resonator with complex dissipative behavior. It is shown that it is possible to raise the quality factor by introducing additional absorption at selected frequencies.

In recent years, a new line of research has appeared on suppressing cascade processes involving energy pumping up the spectrum, which leads to effective absorption of strong waves in nonlinear nondispersive media. This applies particularly to the evolution of nonlinear sound in a medium containing a resonant absorber [1-3]. It has been found that in the simplest case, additional absorption at the second harmonic increases the discontinuity generation length [3]. In the more general case, selective absorption at a series of combination frequencies enables one to raise the performance in a given parametric process. In [3], a mathematical technique was devised for calculating wave interactions in a medium containing a distributed absorber. However, it proved impossible to obtain an exact analytic description of wave evolution in a free half-space. Also, there is the residual open problem of the practical design of nonlinear media with given dissipation and dispersion.

Here we consider the scope for controlling energy flow along the spectrum in the establishment of nonlinear oscillations in an acoustic resonator. It is found that any spatially localized input to the acoustic-wave spectrum under certain conditions plays the same part as with a continuously distributed input in wave propagation in a half-space. Therefore, in some cases one can avoid the problem of creating a medium with the necessary bulk properties by using a cavity with certain properties at the boundary. There is a difference from the case of free propagation [3] in that physical significance attaches to stationary waves in a resonator, which enables one to simplify the treatment and to progress further in the exact analytic description of nonlinear wave processes in media with complex dissipation.

We now examine the establishment of nonlinear oscillations in a resonator having rigid boundaries lying in the planes $x' = 0, L$ (L is cavity length and x' is the spatial coordinate). For definiteness, let the resonator be excited by harmonic oscillations at the ideally reflecting boundary $x' = 0$:

$$v(x'=0) = v_1 \sin \omega t. \qquad (1)$$

Here v_1 is the amplitude of the vibrational velocity and ω is frequency. There is a frequency-selective effect on the waves propagating in the resonator when they interact with the boundary x' = L, which has a frequency-dependent reflection coefficient R and which may also perform oscillations of complex form $v_L(\omega t)$ with the period of the fundamental vibrations. That is, the vibration spectrum of the boundary x' = L contains only components nω (n is an integer).

The nonlinear-wave profile stabilization can be represetned as follows. At time t = 0, a piston in the plane x' = 0 begins to radiate waves. The evolution of the wave profile during propagation to x' = L is described by the equation for a simple wave [4]:

$$\frac{\partial v}{\partial x}-\frac{\varepsilon}{c_0^2}v\frac{\partial v}{\partial \tau}=0, \tag{2}$$

where ε is the nonlinear parameter, c_0 is the speed of sound, x is the spatial coordinate in the propagation direction, and $\tau = t - x/c_0$ is the accompanying coordinate. On reflection from the boundary x' = L, the spectrum components $\hat{v}^{(n)}$ change stepwise:

$$\hat{v}^{(n)}(x=L+0)=R_n\hat{v}^{(n)}(z=L-0)-\hat{v}_L^{(n)}. \tag{3}$$

Here $R_n = R(n\ )$ is the reflection coefficient for harmonic n and $\hat{v}_L^{(n)}$ are the spectral components of the vibrational velocity of boundary x' = L. We note that x is the coordinate along which one reckons the distance traveled by the wave; x = x' only for 0 < x' < L. Equation (3) applies for the sine and cosine components separately:

$$\begin{aligned}\widehat{v}_s^{(n)}&=\frac{1}{\pi}\int_0^{\pi} v(x,t)\sin n\omega t\cdot d(\omega t),\\ \widehat{v}_c^{(n)}&=\frac{1}{\pi}\int_0^{\pi} v(x,t)\cos n\omega t\cdot d(\omega t),\\ v&=\sum_{n=1}^{\infty}[\widehat{v}_s^{(n)}(x)\sin n\omega t+\widehat{v}_c^{(n)}(x)\cos n\omega t].\end{aligned} \tag{4}$$

As the waves in opposite directions do not interact in this weak-wave approximation [4], the subsequent propagation to the boundary x' = 0 (x = 2L) is described by (2). On reflection from this boundary, only the sine component at the fundamental increases stepwise by virtue of (1):

$$v(x=2L+0)=v(x=2L-0)+v_1\sin\omega t. \tag{5}$$

The process then repeats.

Consequently, the wave profile evolution is described by the following equations in a coordinate system in which the wave direction does not alter:

$$\begin{gathered}\frac{\partial v}{\partial x}-\frac{\varepsilon}{c_0^2}v\frac{\partial v}{\partial \tau}=0,\ Lm<x<L(m+1),\ m=0,1,2,\ldots,\\ v(x=2Lm+0)=v(x=2Lm-0)+v_1\sin\omega t,\\ \widehat{v}^{(n)}[x=L(2m+1)+0]=R_n\widehat{v}^{(n)}[x=L(2m+1)-0]-\widehat{v}_L^{(n)}\end{gathered} \tag{6}$$

with the boundary conditions v(x < 0) = 0. The latter relation in (6) is convenient to rewrite in the following form by using the definition of (4):

$$v[x = L(2m+1)+0] = v[x = L(2m+1)-0] - v_L(\omega t) - \\ -\sum_{n=1}^{\infty}(1-R_n)\{\widehat{v}_s^{(n)}[x = L(2m+1)-0]\sin n\omega t + \\ + \widehat{v}_c^{(n)}[x = L(2m+1)-0]\cos n\omega t\}. \quad (7)$$

We use the properties of the δ function to reduce system (6) and (7) to one equation containing sources:

$$\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial \tau} = v_1 \sin \omega t \sum_{m=0}^{\infty}\delta(x-2Lm) - \Big\{v_L(\omega t) + \\ + \sum_{n=1}^{\infty}(1-R_n)[\widehat{v}_s^{(n)}\sin n\omega t + \widehat{v}_c^{(n)}\cos n\omega t]\Big\}\sum_{m=0}^{\infty}\delta[x - L(2m+1)]. \quad (8)$$

Near linear resonance, $L=\pi(k+\Delta)c_0/\omega$, where k is an integer, and $|\Delta|\ll 1$, and one can use the periodicity of the functions in time to perform the transformation $\sin\omega t \to \sin(\omega\tau+\pi\Delta x/L)$, $v_L(\omega t) \to v_L(\omega\tau+\pi k+\pi\Delta x/L)$, $\cos n\omega t \to (-1)^{nk}\cos n(\omega\tau+\pi\Delta x/L)$, etc. It is convenient to introduce the following dimensionless variables: $v=v/v_0$, $\tau=\omega\tau$, $x=x/l$, where $v_0=v_1(2l_p/L)^{1/2}$ is the characteristic stationary-wave amplitude in the resonator with an ideal immobile boundary at x' = L, $l_p=c_0^2/(\varepsilon\omega v_1)$ is the length over which discontinuity is formed in the wave radiated by the wall x' = 0, while l is a certain characteristic distance over which the wave profile beomes substantially distorted. Then (8) becomes

$$\frac{\partial v}{\partial x} - \frac{l}{l_p}\left(\frac{2l_p}{L}\right)^{1/2} v\frac{\partial v}{\partial \tau} = \left(\frac{L}{2l_p}\right)^{1/2}\sin\tau' \sum_{m=0}^{\infty}\delta\left(x - 2\frac{L}{l}m\right) - \\ - \Big\{v_L(\tau'+\pi k) + \sum_{n=1}^{\infty}(1-R_n)[\widehat{v}_s^{(n)}\sin n\tau' + \widehat{v}_c^{(n)}\cos n\tau']\Big\} \times \\ \times \sum_{m=0}^{\infty}\delta\left[x - \frac{L}{l}(2m+1)\right], \quad (9)$$

$$\tau' \equiv \tau + \pi\Delta x/L.$$

Here and subsequently, $v_s^{(n)}$ and $v_c^{(n)}$ are defined by (4), in which the substitutions $\omega t \to \tau$; $(x, t) \to (x, \tau)$ are made. If the distortion over the cavity length is small:

$$l \gg L, \quad (10)$$

one can transfer on the right in (9) from summation with respect to m to integration with respect to $2mL/l$ and get the following simplified equation:

$$\frac{\partial v}{\partial x} - \frac{l}{l_p}\left(\frac{2l_p}{L}\right)^{1/2} v\frac{\partial v}{\partial \tau} = \frac{l}{2L}\left(\frac{L}{2l_p}\right)^{1/2}\sin\tau' - \\ - \frac{l}{2L}\Big\{v_L(\tau'+\pi k) + \sum_{n=1}^{\infty}(1-R_n)[\widehat{v}_s^{(n)}\sin n\tau' + \widehat{v}_c^{(n)}\cos n\tau']\Big\}. \quad (11)$$

Then the set of δ sources can be replaced by sources of finite intensity in describing the profile varying slowly over distances of the order of L, these sources being continuously distributed in space. From the physical viewpoint, one can say that if condition (10) is obeyed, the excitation by the wall vibrations is similar to the generation of sound in a free half-space by distributed sources moving with transonic speed [4,5] (the first two terms on the right in (11)). Comparison of (11) with the equations of [3,6] shows that the use of a nonideal boundary ($R_n \neq 1$) is equivalent to providing a medium whose absorption coefficient for harmonic n is $\gamma_n=(1-R_n)/2L$ (the other terms on the right). We note that (11) can be derived by a more formal mathematical method [4], but the derivation given here has considerable physical clarity.

As an example, we examine the scope for increasing the amplitude of the fundamental and the quality factor by selective action on the second harmonic. We first examine the effects of resonant absorption of the second harmonic on energy redistribution over the spectrum. By virtue of (11), the profile of a stationary wave ($x \to \infty$) in the case $v_L=0$, $\Delta=0$, $R_n=1$ satisfies the following for $n \neq 2$:

$$v\frac{\partial v}{\partial \tau}=-\frac{1}{4}\sin\tau+\frac{l_d}{l_a}\left(\frac{L}{2l_p}\right)^{1/2}\hat{v}_s^{(2)}\sin 2\tau. \qquad (12)$$

Here l_a^{-1} is the absorption coefficient for the second harmonic, which may incorporate bulk absorption: $l_a^{-1}=\gamma_2+\gamma_2^V$, where γ_2^V is the bulk absorption coefficient for the second harmonic [3,6].

The solution to (12) that satisfies the conditions for the mean value of v to be zero over the piston period [7,8] and the absence of shock negative-pressure waves takes the form

$$v=\operatorname{sign}(\tau-2\pi p)\left|\cos\frac{\tau}{2}\right|\left(1+\Pi\hat{v}_s^{(2)}\sin^2\frac{\tau}{2}\right)^{1/2},$$
$$(2p-1)\pi\leqslant\tau\leqslant(2p+1)\pi,\quad p=0,\ \pm1,\ \pm2,\ \pm\ldots \qquad (13)$$

Then in terms of our variables, the wave profile is dependent on the single dimensionless parameter $\Pi=4(2l_dL/l_a^2)^{1/2}$. We use (4) and the Fourier transform of (13) for n = 2 to get an inexplicit dependence of the second-harmonic amplitude $\hat{v}^{(2)}\equiv\hat{v}_s^{(2)}$ on parameter Π:

$$\hat{v}^{(2)}=\frac{1}{\pi(\Pi\hat{v}^{(2)})^2}\left[\frac{(1+\Pi\hat{v}^{(2)})^2}{(\Pi\hat{v}^{(2)})^{1/2}}\arcsin\left(\frac{\Pi\hat{v}^{(2)}}{1+\Pi\hat{v}^{(2)}}\right)^{1/2}-\frac{5}{3}\Pi\hat{v}^{(2)}-1\right]. \qquad (14)$$

When (14) has been used to derive $\hat{v}^{(2)}=\hat{v}^{(2)}(\Pi)$ (Fig. 1), the solution of (13) explicitly describes the nonlinear-wave profile in a medium having selective absorption for various values of Π (Fig.2). The spectral expansion enables one to determine the amplitude of any component, for example:

$$\hat{v}^{(1)}=\frac{4}{3\pi}+\frac{\Pi(\hat{v}^{(2)})^2}{2},\quad \hat{v}^{(3)}=\hat{v}^{(2)}+\frac{5}{2\Pi}-\frac{4}{3\pi\Pi\hat{v}^{(2)}}. \qquad (15)$$

Figure 2 shows the amplitudes of the first three harmonics and the quality factor $Q/Q(\Pi=0)=1+(\Pi\hat{v}^{(2)})/4$ as functions of Π constructed by means of (14) and (15). Then as the second-harmonic absorption increases, the amplitude of the fundamental and the quality factor increase, while the amplitudes of the other spectral components fall. In the case $\Pi \gg 1$:

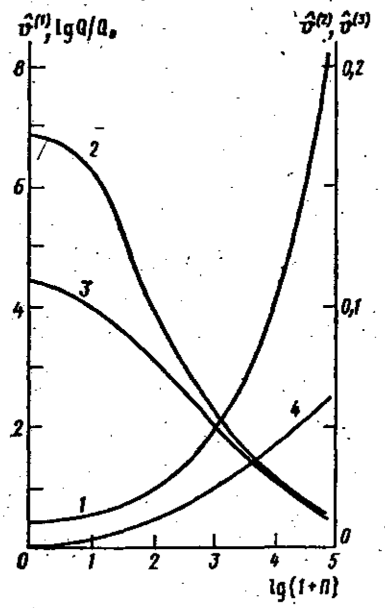

Fig. 1

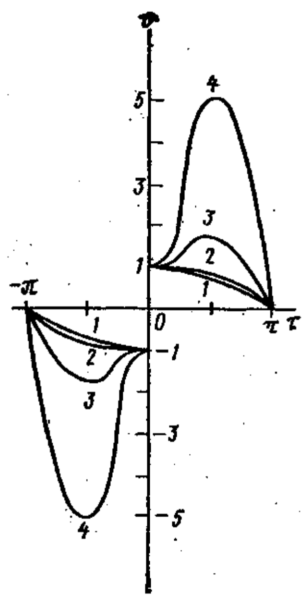

Fig. 2

Fig. 1. Dependence of the dimensionless harmonic amplitudes and quality factor on Π: 1 — $\hat{v}^{(1)}$, 2 — $\hat{v}^{(2)}$, 3 — $\hat{v}^{(3)}$, 4 — $\lg Q/Q_0$.

Fig. 2. Stationary traveling-wave profiles: Π = 0 (1); 6.4 (2); 105 (3) and 2380 (4).

$$v \simeq \frac{1}{2}\left(\frac{\Pi}{2}\right)^{1/3}\sin\tau,\quad \hat{v}^{(2)} \simeq \hat{v}^{(3)} \approx \frac{1}{2}\left(\frac{\Pi}{2}\right)^{-1/3},\quad \frac{Q}{Q_0} = \frac{1}{4}\left(\frac{\Pi}{2}\right)^{2/3}. \qquad (16)$$

We thus have established that additional second-harmonic absorption opens up a new energy-loss channel, but on the other hand so greatly hinders the energy transfer up the spectrum (where dissipation mainly occurs in the upper part) that the energy accumulated in the cavity actually increases.

These results are accurate if condition (10) used in deriving (11) is obeyed. In the above problem, l is the characteristic distance over which a wave of the form of (13) may be substantially distorted in free propagation. Mathematically, l may be estimated as the length for the formation of a discontinuity in a harmonic wave of appropriate amplitude. Then (10) sets an upper bound to Π for which the asymptote of (16) applies: $\Pi \ll (l_d/L)^{3/2}$.

Another application of this technique is active second-harmonic quenching in the nonlinear wave. This can be attained if the boundary x' = L radiates the second harmonic in antiphase to that generated by the nonlinear distortion of the wave radiated from x' = 0. We put $x \to \infty,\ R_n = 1,\ v_L(\omega t) = v_2 \sin 2\omega t$ in (11) to get

$$v\frac{\partial v}{\partial \tau} = -\frac{1}{4}\sin\tau + \frac{1}{2}\left(\frac{l_d}{2L}\right)^{1/2} v_2 \sin 2\tau. \qquad (17)$$

Then (17) shows that the amplitude of the fundamental increases for $v_2 > 0$, as is also indicated by the mathematical analogy between (12) and (17). This can be treated as parametric amplification of the fundamental in the field of the second-harmonic wave [9]. We note that the sources on the right in (17) are independent of the wave parameters, i.e., are given, in contrast to (12). This enables one to examine the parametric interaction in the resonator under nonstationary conditions and with an arbitrary phase shift between the oscillations of the boundaries [9].

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