A COLORED-FERMION TRIPLET IN THE FIELD OF A POLARIZED YANG-MILLS WAVE

V. R. Khalilov and V. C. Perez-Fernandez

Vestnik Moskovskogo Universiteta. Fizika,
Vol. 39, No. 6, pp. 35-38, 1984

UDC 539.12.01

An exact solution has been derived to Dirac's equation in a plane-wave Yang-Mills field of special configuration.

1. PLANE-WAVE SOLUTIONS TO THE YANG-MILLS EQUATION

The classical Yang-Mills field equations for group SU(2) take the form

$$\partial^{\mu} F_{\mu\nu}{}^{a} + g\varepsilon^{abc} A^{b\mu} F_{\mu\nu}{}^{c} = 0,$$
$$F_{\mu\nu}{}^{a} = \partial_{\mu} A_{\nu}{}^{a} - \partial_{\nu} A_{\mu}{}^{a} + g\varepsilon^{abc} A_{\mu}{}^{b} A_{\nu}{}^{c}, \tag{1}$$

where ε^{abc} are the structural constants of group SU(2), which in this case coincide with the unit absolutely antisymmetric third-rank tensor ε^{ijk}, while g is the interaction constant (in (1), g is the constant characterizing the self-action), $F_{\mu\nu}{}^{a}$ is the tensor for the strength of the Yang-Mills field, which is an analog of the electromagnetic-field tensor, A^a are the 4-potentials of the Yang-Mills field, and a, b, c = 1, 2, 3 are indices that enumerate the components of the field functions in isotopic space. Equations of motion (1) are nonlinear. Nontrivial plane-wave solutions to (1) that are analogs of planar electromagnetic waves have been derived in [1].

In the Lorentz calibration,

$$\partial_{\mu} A^{a\mu} = 0, \qquad k_{\mu} A^{a}{}_{\mu} = 0$$

these solutions can be written as

$$A_{\mu}^{1} = n_{\mu}^{(1)} f_1(\varphi),\ A_{\mu}^{2} = n_{\mu}^{(2)} f_2(\varphi),\ A_{\mu}^{3} = \frac{k_{\mu}}{g}\frac{d}{d\varphi}\operatorname{arctg}\frac{f_1}{f_2}, \tag{2}$$

where the functions $f_1(\phi)$, $f_2(\phi)$ are arbitrary functions of the phase $\phi = kx$, where k is an arbitrary constant isotropic 4-vector ($k^2 = 0$), and n is a spacelike 4-vector orthogonal to the k vector ($k \cdot n = 0$). The Greek letters μ and ν enumerate the components of the tensors in the Minkowski space with metric tensor $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

The 4-potentials of (2) correspond Yang-Mills field strengths

$$F_{\mu\nu}^{1,2} = (k_\mu n_\nu - k_\nu n_\mu) f_{1,2} \frac{d}{d\varphi} \ln \sqrt{f_1^2 + f_2^2},$$
$$F_{\mu\nu}^{3} = 0. \tag{3}$$

We note that the quantities of (3) are nonlinear functions of the potentials. It is readily shown that the energy-momentum tensor for the Yang-Mills field takes the form

$$T_{\mu\nu} = k_\mu k_\nu (f_1^2 + f_2^2) \left[\frac{e}{d\varphi} \ln \sqrt{f_1^2 + f_2^2} \right]^2 = k_\mu k_\nu \Phi(\varphi),$$

where $T_{00} > 0$ is an essentially positive quantity and $T_\mu{}^\mu = 0$, since $k^2 = 0$, as should be the case for a massless field. It is evident that $T_{\mu\nu}$ satisfies the continuity equation

$$\partial^\mu T_{\mu\nu} = 0,$$

since

$$\partial^\mu T_{\mu\nu} = k^\mu k_\mu k_\nu \Phi(\varphi).$$

Consequently, $T_{\mu\nu}$ is a locally conserved symmetrical tensor transverse in the 4-dimensional sense.

In [2], the concept of a covariantly constant field was introduced. The strengths of such fields obey the equations

$$\nabla_\rho F_{\mu\nu}{}^a = (\delta^{ab}\partial_\rho + g\varepsilon^{abc} A_\rho{}^c) F_{\mu\nu}{}^b = 0. \tag{4}$$

Covariantly constant Yang-Mills fields represent a natural generalization of constant homogeneous magnetic fields, for which the condition of constancy may be written as

$$\partial_\rho F_{\mu\nu} = 0.$$

A substantially new situation arises from the nonlinearity of (4). The nonlinearity of the Yang-Mills equations and (4) causes the color components of the potentials to couple (self-action), and this leads to an extension of the class of covariantly constant fields, in contrast to the abelian case.

Consider the field of (3). The functions f_1 and f_2 may be chosen in various ways to satisfy (4). Let $f_1 = f_0 \varphi$, $f_2 = f_0 \varphi$, $f_0 = \text{const}$. In that case, the field strengths

$$F_{\mu\nu}{}^{1,2} = (k_\mu n_\nu - k_\nu n_\mu) f_0 \tag{5}$$

are not dependent on x_μ at all, and the corresponding fields are constant homogeneous crossed color ones. However, the condition for covariant constancy of the field of (3) takes the form

$$\left[\frac{d}{d\varphi} \ln P \right]^2 + \frac{d^2}{d\varphi^2} \ln P = 0,$$
$$P = \sqrt{f_1^2 + f_2^2}, \quad \varphi = k \cdot x.$$

Consequently, the field of (3) is covariantly constant if

$$P = f_0 \varphi, \quad f_0 = \text{const}.$$

Therefore, the potentials of (2) describe a covariantly constant field if we take

$$f_1 = f_0\varphi \cos\varphi, \qquad f_2 = f_0\varphi \sin\varphi.$$

In that case, the solutions

$$F_{\mu\nu}^{1,2} = (k_\mu n_\nu - k_\nu n_\mu) f_0 \{\cos\varphi, \ \sin\varphi\} \tag{6}$$

may be treated as planar linearly polarized waves propagating with the speed of light with constant field amplitudes. The phases of these color waves are displaced by $\pi/2$. We note that the strengths of the covariantly constant field are linear in the amplitude of the 4-potential f_0. We show below that the action of the fields of (5) and (6) on quarks leads to physically equivalent consequences, since a major part of the quark wave function is dependent only on the combination f_1^2 and f_2^2.

2. A FERMION TRIPLET IN AN EXTERNAL COLORED PLANE-WAVE FIELD

We consider a triplet of colored fermions (quarks)

$$\Psi = (\Psi^A, \Psi^B, \Psi^C)$$

in the external field of (2) and (3). Within the field-theory approach, the quarks interact via the gluon field, which in the general case is an 8-component field in color space (it realizes the adjoint representation of group $SU_c(3)$). Since, however, in what follows we consider the field of (2) as a classical field, we specify that $A_\mu^a = 0$ for $a = 4, 5, 6, 7, 8$.

The equations of motion for quarks in an external color field take the form [3]

$$i\gamma^\mu \left(\partial_\mu - \frac{ig}{2} A_\mu^a \lambda^a\right) \Psi - m\Psi = 0, \tag{7}$$

where γ^μ are 3-block Dirac matrices and λ^a are Gell-Mann matrices. We substitute the potentials of (2) into (7) to get that the field of (2) does not act on the component ψ^C, and ψ^C is described by a free Dirac equation. We further assume that $\sqrt{f_1^2 + f_2^2} \neq 0$, and introduce $\theta = \operatorname{arctg}(f_2/f_1)$, which enables us to show that the equation system for the components Ψ^A and Ψ^B is consistent if Ψ^A and Ψ^B are related by a phase factor

$$\Psi^A = e^{-i\theta} \Psi^B.$$

Then the quark equation of motion is equivalent to the Dirac equation for an electron in the field of a planar electromagnetic wave:

$$\left(\hat{\partial} - \frac{ig}{2} \hat{A}\right) \Psi - m\Psi = 0, \tag{8}$$

where

$$A_\mu = -\frac{k_\mu}{g} f_3 \mp n_\mu \sqrt{f_1^2 + f_2^2}.$$

The solution to (8) can be written out at once from the results of [4]:

$$\Psi = \left[1 - \frac{g}{4(p\cdot k)}\hat{k}\hat{A}\right]\frac{u}{\sqrt{2p_0}}e^{iS},$$

$$S = -(p\cdot x) + \frac{g}{2(p\cdot k)}\int\left[(p\cdot A) + \frac{g}{4}(A\cdot A)\right]d\varphi,$$

with a bispinor u satisfies

$$(\hat{p} - m)u = 0.$$

In explicit form we have for Ψ^A and Ψ^B that

$$\Psi^A = Q\exp\left\{i\tilde{S} + \frac{i}{2}\int f_3 d\varphi\right\},$$

$$\Psi^B = Q\exp\left\{i\tilde{S} - \frac{i}{2}\int f_3 d\varphi\right\},$$

where

$$Q = \left[1 + \frac{g\sqrt{f_1^2+f_2^2}}{4(p\cdot k)}\hat{n}\hat{k}\right]\frac{u}{\sqrt{2p_0}},$$

$$\tilde{S} = -(p\cdot x) + \frac{lg}{2(p\cdot k)}\int\left[(p\cdot n) - \frac{g}{4}\sqrt{f_1^2+f_2^2}\right]\sqrt{f_1^2+f_2^2}\,d\varphi.$$

The operator $i\partial_\mu + \frac{g}{2}\lambda^a A^a_\mu$, here acts as the kinetic momentum operator, and its action on the quark states Ψ is defined as follows:

$$\left(i\partial_\mu + \frac{g}{2}A^a_\mu\lambda^a\right)\Psi^{A,B} = \left[\pi_\mu + \frac{igk_\mu}{4(p\cdot k)}(\sigma\cdot F)\right]\Psi^{A,B}, \qquad (9)$$

where

$$\pi_\mu = p_\mu - \frac{g}{2(p\cdot k)}k_\mu\left[(p\cdot n) - \frac{g}{4}\sqrt{f_1^2+f_2^2}\right]\sqrt{f_1^2+f_2^2} + \frac{g}{2}n_\mu\sqrt{f_1^2+f_2^2},$$

$$\sigma \equiv \sigma^{\rho\beta} = \frac{1}{2}(\gamma^\rho\gamma^\beta - \gamma^\beta\gamma^\rho),\ \rho < \beta,$$

$$F \equiv F_{\rho\beta} = \frac{k_\beta n_\rho - k_\rho n_\beta}{2}\sqrt{f_1^2+f_2^2}\frac{d}{d\varphi}\ln\sqrt{f_1^2+f_2^2}.$$

The second term on the right in (9) characterizes the interaction of the quark spin with the color field. We see that the effective quark spin interacts with the colorless field proportional to $\sqrt{f_1^2+f_2^2}$.

The density of the energy-momentum tensor for the quark field $T_{\mu\nu}(x)$ is written in a form analogous to the expression for the density of the energy-momentum tensor for an electron in the field of an electromagnetic wave:

$$T_{\mu\nu}(x) = \overline{\Psi}\gamma_\mu\Psi\pi_\nu + \frac{ig}{2(p\cdot k)}k_\nu\overline{\Psi}\gamma^5\gamma^\lambda\Psi F^*_{\lambda\mu},$$

where

$$F^*_{\lambda\mu} = \frac{1}{2}\varepsilon_{\lambda\mu\nu\rho}F^{\nu\rho}$$

is a tensor dual to $F_{\lambda\mu}$.

In a similar fashion, we define the 4-pseudovector for the quark spin s_λ

(s_λ is not an integral of motion):

$$i\overline{\Psi}\gamma_5\gamma_\lambda\Psi = s_\lambda\overline{\Psi}\Psi.$$

One can determine the motion of the quark spin in the field of a colored wave by calculating the element $\overline{\Psi}\gamma_5\gamma_\lambda\Psi$ and introducing the conserved 4-pseudovector s_0 for state u:

$$s_\lambda = s_{0\lambda} + \frac{g}{2(p\cdot k)}[B_\lambda(k\cdot s_0) - k_\lambda(B\cdot s_0)] - \frac{g^2(B\cdot B)}{8(p\cdot k)^2}(s_0\cdot k)k_\lambda,$$

where

$$B_\lambda = n_\lambda\sqrt{f_1^2 + f_2^2}.$$

In conclusion we note that one can classify further the quark states in field of (3) by analogy with the classification of the states of an electron in the field of an electromagnetic wave.

REFERENCES

1. S. Coleman, Phys. Lett., vol. 70B, no. 1, p. 59, 1977.
2. I. A. Batalin, S. G. Matinyan, and G. K. Savvidi, Yad. Fiz., vol. 26, p. 407, 1977.
3. I. V. Andreev, Chromodynamics and Rigid Processes at High Energies [in Russian], Nauka, Moscow, 1981.
4. V. I. Ritus, Trudy FIAN, vol. III, pp. 5-151, Nauka, Moscow, 1979.

4 January 1984

Theoretical Physics Department