THE ACCURACY IN POTENTIAL RECOVERY IN AN INVERSE SCATTERING PROBLEM ($l \geq 1$)

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The paper discusses the error in recovering potentials within the framework of exactly soluble quantum models for the case $l \geq 1$. It is shown that even for $l = 1$, one can obtain reliable information on the behavior of the potential in this way only at large distances.

In this note we consider the error with which one can recover a potential in the case $l \geq 1$ due to the lack of data on scattering above the energy N^2 if the phases are known with a certain error within the energy range $[0, N^2]$ (it is assumed that there are no bound states).

We consider two boundary-value problems with l singularities corresponding to the radial part of the Shroedinger equation:

$$\frac{d^2}{dx^2}\Psi_l^{(i)}(x) - \left[V^{(i)}(x) + \frac{l(l+1)}{x^2}\right]\Psi_l^{(i)}(x) + k^2\Psi_l^{(i)}(x) = 0, \tag{1}$$

$$\Psi_l^{(i)}(x)\underset{x\to 0}{=}0. \tag{2}$$

The conditions imposed on $V^{(i)}(x)$, are analogous to those introduced in [1]. We have to establish how greatly the potentials $V^{(1)}$ and $V^{(2)}$ may differ for (1) and (2) if the scattering phases lie within the error limits. For this purpose, the analogy with the case $l = 0$ [1] enables us to establish the correctness of the formula

$$\Delta V_l(x) = -\frac{1}{\pi}\int_N^\infty \Delta S_l(k)\frac{d}{dx}[f_l^{(1)}(k,x)f_l^{(2)}(k,x)]\,dk, \tag{3}$$

where

$$\Delta S_l(k) = S_l^{(1)}(k) - S_l^{(2)}(k), \quad \Delta V_l(x) = V_l^{(1)}(x) - V_l^{(2)}(x),$$

and $f_l^{(i)}(k, x)$ is the Jost solution for the corresponding boundary-value problems. We used the Kram-Krein transformations [2] for equations of Sturm-Liouville type:

$$f_l(k,x) = \frac{iW[f_{l+1}(k,x), z_{l+1}(x)]}{kz_{l+1}(x)}, \tag{4}$$

$$f_l(k, x) = \frac{iW[f_{l-1}(k, x), y_{l-1}(x)]}{ky_{l-1}(x)}, \tag{5}$$

where $y_l(x)$, $z_l(x)$ are the regular and irregular solutions to (1) for $k = 0$ with asymptotes at infinite of x^{l+1} and x^{-l}, correspondingly. We use the first transformation of (4) and the identity

$$\frac{d}{dx}\frac{W[\Psi(k_1, x), \Psi(k_2, x)]}{k_1^2 - k_2^2} = \Psi(k_1, x)\Psi(k_2, x), \tag{6}$$

which applies for any pair of solutions to (1) for $k_1 \neq k_2$, to obtain the following equation (the arguments to the Jost solutions are temporarily omitted):

$$\frac{d}{dx}[f_l^{(1)} f_l^{(2)}] = ik[f_{l+1}^{(1)} f_l^2 + f_{l+1}^{(2)} f_l^{(1)}] - Z_{l+1}(x)[f_l^{(1)} f_l^{(2)}]. \tag{7}$$

Similarly, from (5) and (6) we have

$$\frac{d}{dx}[f_{l+1}^{(1)} f_{l+1}^{(2)}] = ik[f_l^{(1)} f_{l+1}^{(2)} + f_l^{(2)} f_{l+1}^{(1)}] - Y_l(x)[f_{l+1}^{(1)} f_{l+1}^{(2)}], \tag{8}$$

where we have used the symbols

$$Z_l(x) = \frac{d}{dx}\ln[z_l^{(1)}(x) z_l^{(2)}(x)]; \quad Y_l(x) = \frac{d}{dx}\ln[y_l^{(1)}(x) y_l^{(2)}(x)]. \tag{9}$$

We further assume that all the potentials $V_l^{(i)}$ behave identically for the different l as $x \to 0$ and $x \to \infty$. In that case, Marchenko's theorem [2] applies, according to which $S_l(k)$ for the potential $V_l^{(i)}(x)$ is also an S matrix $S_{l_1}(k)$ for a certain potential $V_{l_1}^{(i)}(k)$ $(l_1 \neq l)$, i.e.,

$$S_{l+1}(k) = S_l(k). \tag{10}$$

Then (3), (10) and (8), (9) imply the basic formula

$$\Delta V_{l+1}(x) - \Delta V_l(x) = Y_l(x)\int_x^\infty \Delta V_{l+1}(t)\,dt - Z_{l+1}(x)\int_x^\infty \Delta V_l(t)\,dt. \tag{11}$$

This equation enables one to express the error in constructing $V_l(x)$ from the l phase via the set $\{\int_x^\infty \Delta V_{l_1}(t)\,dt\}_{l_1=0,1,\ldots,l}$ and $\Delta V_0(x)$, where $V_{l_1}(x)$ is the potential for the boundary-value problem of (1) and (2) corresponding to various l_1, where the scattering data used in constructing $V_{l_1}(x)$ have been taken as $S_l(k)$ for all l_1. For example, (11) appears as follows for the case $l = 1$:

$$|\Delta V_1(x)| \leqslant |\Delta V_0(x)| + |Z_0(x)\int_x^\infty \Delta V_1(t)\,dt| + |Z_1(x)\int_x^\infty \Delta V_0(t)\,dt|. \tag{12}$$

We note that all the integrals on the right in (12) may be estimated by means of (4) and (5) in the case $l \leq 1$ or by means of similar transformations considered in [3] if $l > 1$. For example, we use representation (6) from [4] and the bound $|f_0^{(i)}(k, x)| \leqslant \exp[\gamma(x, N)]$ [1] to get

$$\left|\int_x^\infty \Delta V_0(t)\,dt\right| \leqslant \frac{|\Delta S(k)|}{\pi}\,\frac{\alpha^2(x)\gamma(x, N)}{N}.$$

Here we have used the same symbols as in [4]; we have also introduced the additional definition

$$\gamma(x, N) = \exp\left[\int_x^{x+1/N} \alpha(t)\,dt\right] + \exp\left[2\int_x^{x+1/N} \alpha(t)\,dt\right].$$

We see from (12) that the accuracy in recovering the potential from p-scattering data is always much worse than that for s waves. We estimate this additional error introduced by the centrifugal barrier by means of the last term in (12). Here the majorant for $Z_1(x)$ is readily found by considering the corresponding integral equation for $\alpha_1(x) = \int_x^\infty \alpha(t)\,dt < 3\ln 2$:

$$|Z_1(x)| \leqslant \frac{2}{x}\left(\frac{3e^{\alpha_1(x)/3} - 2}{2 - e^{\alpha_1(x)/3}}\right).$$

As a result, from the inequality $|\Delta S(k)| \leqslant 2$ we get

$$|\Delta V_1(x)| \leqslant |\Delta V_0(x)| + \frac{8}{\pi}\frac{\alpha^2(x)\gamma(x, N)}{Nx}\left(\frac{3e^{\alpha_1(x)/3} - 2}{2 - e^{\alpha_1(x)/3}}\right),$$

where the bound of (8) from [4] applies for $|\Delta V_0(x)|$.

We see that the error increases considerably for x small, which is physically clear. Therefore, the screening by the centrifugal barrier is so strong that it is impossible to recover the structure of the potential at short distances.

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