ONE PROBLEM OF STRUCTURAL DIAGNOSTICS

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A one-dimensional model of the problem about determining the structural characteristics of a medium using an ultrasonic probing method is examined. The question about the uniqueness of solution of the problem is investigated. Asymptotic analysis is used to establish which types of characteristics of the structure may be acquired by the cited method in the high frequency range with an assigned experimental precision.

1. Ultrasonic probing is an important method for nondestructive diagnosis of the internal structures of different articles and living tissues [1]. In this case the internal structure which is characterized by the distribution of a certain parameter $\beta(x)$ as a function of coordinates is judged from the acoustical impedance of the medium $I(\omega)$, measured in the high frequency sonic range. Such a problem is an inverse problem, and Tikhonov regularizing algorithms may be used for its solution [2].

The following questions are fundamental for developing the regularizing algorithms: a) Is the correspondence of $I(\omega) \to \beta(x)$ unambiguous within the model which meets the conditions of the experiment? and b) To what degree, with the assigned experimental precision, may it be counted on to produce detailed information about the structure? These questions are examined in this article for a one-dimensional model which corresponds to the problem of acoustical tomography [1]. They can be answered using asymptotic ($\omega \to \infty$) analysis.

2. Introducing dimensionless variables, the amplitude of the sonic pressure $u(x)$ with harmonic excitation in a one-dimensional medium may be described by the following conditions:

$$u''+\lambda^2\beta(x)u=0,\ 0<x<1,\ u(0)=0,\ u(1)=1, \quad (1)$$

where with evenly small absorption $\beta(x)=\beta_1(x)+i\varepsilon$, $\varepsilon=\text{const}>0$ [3], $\beta_1(x)>0$, $\lambda^2=\omega^2 l^2\rho_0\beta_0$, l is the characteristic length of the system, ρ_0 is the scale of density, ω' is the characteristic probing frequency, $\beta_0=(a^2\rho_0)^{-1}$, and a is the speed of sound in the medium. In problems of the examined type $\varepsilon\ll\beta_1(x)$ and the dimensionless parameter is $\lambda^2 \gg l$. (Thus, in problems of medical tomography, $l \approx 5$-10 cm, $\rho_0 = 1$ g/cm^3, $\omega = 2\pi(1.3$-$1.7)\cdot 10^6$ sec^{-1}, $\beta_0 = 0.1\cdot 10^{-10}$ cm$\cdot$sec$^2\cdot$g^{-1}, and $\varepsilon = 0.04$.)

Assuming that the value of ε is assigned, the structure will be characterized by the parameter $\beta_1(x)$ which is equivalent at an assigned density to the "velocity section" [4]. With each function of $\beta_1(x)$ the theoretical value of the impedance which is comparable with the measurements is given by the formula

$$I(\omega) = i\omega\rho_0 l \Big/ \left(\frac{du}{dx}\right)_{x=l}. \qquad (2)$$

The uniqueness of the correspondence of $I(\omega) \to \beta_1(x)$ may be judged by comparing in problem (1) a model with a piecewise constant characteristic of $\beta_1(x)$ with an arbitrary number of layers n. Such a problem was studied in [4] for a layered structure which is in a semispace and with a purely imaginary $\beta(x)$ (the absorbing medium). It turns out that the corresponding analysis technique may also be applied to the examined case.

A grid of $\{x_p\}$, $p=0,\ldots,n+1$ is introduced, assuming $x_0=0$, $x_{k+1}=x_k+d_{n-k}$ $(d_k>0)$, $x_{n+1}=1$. Assume that $\beta_1(x)=\beta_{n+1-p}=\text{const}>0$ at $x \in (x_p, x_{p+1})$. The multitude of structures characterized by the vector $\boldsymbol{\beta}_1=\{\beta_1, \beta_2, \ldots, \beta_{n+1}, d_1, \ldots, d_n\}$ at any assigned values of n and β_1 is called the K_n^1 class.

Theorem 1. Only one vector of β_1 may correspond to the assigned function $I(\omega)$ in the K_n^1 class.

For the sake of demonstration, the totality P of the "effective" parameters is compared to the physical characteristics of the structure. The parameters are:

$$\tau_k=\sqrt{\rho_k/\rho_{k+1}},\quad k=1, 2, \ldots, n,\quad \theta_p=d_p\sqrt{\rho_p},\quad p=1, 2, \ldots, n+1, \qquad (3)$$

where $\rho_p=\beta_p+i\varepsilon$. It is apparent that in the K_n^1 class the unique vector β_1 corresponds the assigned P. It is noted that $\theta_p=\alpha_p+i\delta_p$, where $\delta_p > 0$.

Analogous to [4], it may be ascertained that the following additive representation is legitimate:

$$\varkappa I(\omega) = \tilde{I}(\lambda) = \sum_{p=0}^{n} S_p,\quad S_p = W_p \frac{\tau_p t_p}{1+\gamma_p \tau_p t_p}, \qquad (4)$$

where κ is not a function of the parameters of the structure, with the exception of β_1; $W_0=1, \gamma_0=0, \tau_0=1$, and at $p \neq 0$

$$W_p=\sum_{k=0}^{p-1}\frac{\tau\,(1-t_k^2)}{(1+\gamma_k\tau_k t_k)^2},\quad \gamma_{p+1}=\frac{\gamma_p\tau_p+t_p}{1+\gamma_p\tau_p t_p},\quad t_p=\frac{1-e_{p+1}}{1+e_{p+1}},\quad e_p=\exp(2i\theta_p\lambda).$$

It is substantive here that $e_p=O(e^{-2\delta_p\lambda})$ at $\lambda \to +\infty$, $t_p=1-2e_{p+1}+o(e_{p+1})$, $1-t_p^2=4e_{p+1}+o(e_{p+1})$. Then, it follows from (4) that with sufficiently large values of λ $\tilde{I}(\lambda) = 1+[4\tau_1/(1+\tau_1)-2]e_1+o(e_1)$ and then τ_1 and θ_1 are unambiguously determined by the assigned impedance. It is assumed, based on the induction, that $\tau_1,\ldots,\tau_k$, $\theta_1,\ldots,\theta_k$ are unambiguously determined from $I(\omega)$ and, consequently, the functions $t_0,\ldots,t_{k-1}$,

$\gamma_0, \ldots, \gamma_k, S_0, \ldots, S_{k-1}$. The function $\tilde{T}^{(k)}_{(\lambda)} = \frac{1}{W_k}\left(\tilde{T}(\lambda) - \sum_{p=0}^{k-1} S_p\right)$ is examined. It follows from (4) that at $\lambda \to +\infty$ $\tilde{T}^{(k)}_{(\lambda)} = \left(\frac{4\tau_{k+1}}{1+\tau_{k+1}} - 2\right) e_{k+1} + o(e_{k+1})$ and, consequently, τ_{k+1} and θ_{k+1} are unambiguously determined. The totality P of the effective parameters and, consequently, $\beta_1 \subset K_n{}^1$, is unambiguously determined from the induction and from $I(\omega)$.

It is noted that with a small change in the course of the discussions, the uniqueness of the correspondence $\{I(\omega), \beta_1\} \to \{n, \beta_1\}$ is established.

3. The demonstrated theorem means that with a sufficiently high precision of measurements and with the use of sufficiently precise regularizing algorithms, it is possible to count on acquiring information about the details of the structure. At the same time, the representation of the impedance (4) used in its demonstration does not answer the question of what kind of information this is with a finite experimental error and with probing in a finite frequency range.

To acquire such an answer, it is natural to compare the model in problem (1) with the continuous characteristic $\beta(x)$. It is noted that in the range of physical values used, the dimensionless parameter λ^2 is large: $\lambda^2 \sim 10^3$. This means that the impedance measured at the corresponding frequencies is comparable with a certain asymptotic ($\omega \to \infty$) representation. Formula (4) may be considered as only an "additively asymptotic" formula [5] and the required representation was acquired here relying on the Wentzel-Kramers-Brillouin method, in particular [6].

It may be noted that the determining $I(\omega)$ value in (2) is

$$\left.\frac{du}{dx}\right|_{x=1} = 1 - \lambda^2 \int_0^1 \left.\frac{\partial G(x, s)}{\partial x}\right|_{x=1} \beta(s)\, s\, ds, \tag{5}$$

where $G(x, s)$ is the Green function for the boundary problem (1), and its derivative under the integral may be evaluated using the asymptotic representations of the fundamental solutions of Eq. (1) given in [6].

The following values are introduced for examination:

$$r(s) = \int_0^s \sqrt{\beta(t)}\, dt, \quad \alpha(s) = \frac{i}{8}\left(\frac{5}{4}\frac{\beta'^2(s)}{\beta^{5/2}(s)} - \frac{\beta''(s)}{\beta^{3/2}(s)}\right), \quad R(s) = \int_0^s \alpha(t)\, dt,$$
$$Q_1(s) = \operatorname{Im}\sqrt{\beta(s)}, \quad Q_2(s) = -\operatorname{Re}\sqrt{\beta(s)}; \quad R_k(s) = \int_0^s Q_k(t)\, dt, \quad k = 1, 2. \tag{6}$$

Then, using the above path, the following expression is arrived at:

$$\left.\frac{dG}{dx}\right|_{x=1} = \sqrt[4]{\frac{\beta(1)}{\beta(s)}}\,\frac{\exp\left(i\lambda\left(r(1) + \sum_{k=1}^{2} R_k(s)\right)\right)}{\exp(2i\lambda r(1)) - 1}\left\{\sum_{k=1}^{2} g_k(s) + O(1/\lambda^2)\right\}, \tag{7}$$

where

$$g_k(s) = 1 + \frac{1}{\lambda}\left(R(1) + (-1)^{k+1} R(s) - 2\,\frac{R(1)\exp(2i\lambda r(1))}{\exp(2i\lambda r(1)) - 1}\right).$$

It is noted that in this expression the exponential members in the denominator must not be ignored in terms of unity, since ε_2 is a magnitude of two less than $\beta_1(x)$.

Placing (7) in (5), integrals from the rapidly oscillating functions are found:

$$F_k \equiv \int_0^1 e^{i\lambda\rho(s)} f_k(s)\, ds,$$

where

$$f_k(s) = \frac{g_k(s)\beta(s)s}{\sqrt[4]{\beta(s)}}, \quad \rho(s) = \sum_{k=1}^{2} R_k(s).$$

The asymptotic representation for them acquired through integrations in terms of parts with the same relative precision as in (7) has the appearance

$$F_k = \left(-\frac{i}{\lambda}\frac{f_k(s)}{\rho'(s)} + \frac{1}{\lambda^2}\frac{f_k'(s)}{\rho'^2(s)}\right) e^{i\lambda\rho(s)}\Big|_0^1 + O(1/\lambda^3). \qquad (8)$$

Using (8), it is possible to acquire from (2) and (5) the asymptotic expression for the impedance, which demonstrates the following claim.

Theorem 2. The following formula is valid in a class of structures with sufficiently smooth distribution of $\beta_1(x)$:

$$I(\omega) = i\sqrt{\frac{\rho_0}{\beta_1^0}}\left(\frac{1}{h_1} - \frac{1}{\lambda}\frac{1-h_2}{h_1^2} + O(1/\lambda^2)\right), \qquad (9)$$

where

$$h_1 = -\frac{\beta(1)}{Q_2(1)}\left(1 + i\frac{Q_1(1)}{Q_2(1)}\right)\operatorname{ctg}(\lambda r(1)),$$

$$h_2 = \frac{3}{4}\frac{\beta'(1)}{Q_2^2(1)} + \frac{\beta(1)}{Q_2^2(1)} - \frac{4\beta(1)}{Q_2(1)}\left(i - \frac{Q_1(1)}{Q_2(1)}\right)\frac{R(1)}{\sin^2(\lambda r(1))}.$$

The demonstrated theorem means that in the first asymptotic order the acoustical impedance is only a function of the values

$$\beta(1),\ \beta'(1),\ \int_0^1 \sqrt{\beta(t)}\,dt,\ \int_0^1 \left[\frac{5}{4}\frac{\beta'^2(t)}{\beta^{5/2}(t)} - \frac{\beta''(t)}{\beta^{3/2}(t)}\right] dt.$$

Correspondingly, in the examined range of parameters and with a measurement error of 1% the ultrasonic probing may provide only these values, or the approximation of $\beta(x)$ (for instance, using polynomials) which is unambiguously determined by them.

It may also be noted that an increase in the measurement precision to 0.1% (this corresponds to consideration of the second order members) may produce a somewhat more detailed idea about the structure.

The acquired representation is also found to be useful in formulating an algorithm for calculating the characteristics of the structure, since direct numerical calculation with large values of ω is impossible.

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REFERENCES

1. Ultrasonic Diagnostics. Collection of Scientific Works of the Institute of Applied Physics of the USSR Academy of Sciences [in Russian], Gorkii, 1983.
2. A. N. Tikhonov and V. Ya. Arsenin, Methods for Solving Noncorrect Problems [in Russian], Nauka, Moscow, 1979.
3. N. A. Trapeznikova and V. B. Glasko, Prikladnaya Geofizika, no. 92, p. 57, 1978.
4. V. B. Glasko and Yu. I. Khudak, ZhVM and MF, vol. 20, p. 482, 1980.
5. V. B. Glasko, ZhVM and MF, vol. 10, p. 1456, 1970.
6. M. V. Fedoryuk, Asymptotic Methods for Linear Conventional Differential Equations [in Russian], Nauka, Moscow, 1983.

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