COSMOLOGY IN THE EINSTEIN-CARTAN THEORY. 1. SPIN DYNAMICS

V. A. Korotkii and Yu. N. Obukhov

Vestnik Moskovskogo Universiteta. Fizika,
Vol. 42, No. 6, pp. 3-8, 1987

UDC 520.13:531.51

A study is reported of spin dynamics for a Wayssenhoff liquid in a gravitational field. The following theorem is proved: the nontrivial spin of the Wayssenhoff liquid that satisfies the Frenkel condition is conserved if and only if the liquid moves without acceleration.

The Einstein-Cartan theory is a direct generalization of the general theory of relativity within the framework of the gauge approach in gravitation [1]. The gravitational Lagrangian is constructed in the Einstein-Cartan theory from the Hilbert-Einstein general theory of relativity by replacing the Christoffel symbols with the Riemann-Cartan connectedness (Lorentz gauge field), $L_g=-(1/2\varkappa)R$, $\varkappa=8\pi G/c^4$. The field equations are derived in the Einstein-Cartan theory from a variational principle. In the first-order formalism, when the independent variables are the gravitational gauge fields $h_\mu{}^a$ and $\Gamma^a{}_{b\mu}$, the total action $S=\int d^4x h(L_g+L_m)$ leads to the Einstein-Cartan (-Sciama-Kibble) equations

$$R^\mu{}_a-(1/2)Rh^\mu{}_a=\varkappa T^\mu{}_a, \tag{1}$$

$$Q^\mu{}_{ab}-Q_a h^\mu{}_b+Q_b h^\mu{}_a=\varkappa J^\mu{}_{ab} \tag{2}$$

where the contractions of the local Lorentz curvature $R^a{}_{b\mu\nu}$ and twist $Q^a{}_{\mu\nu}$ (which are the strengths of the gravitational gauge field) are defined in a standard way: $R_{a\mu}=R^b{}_{a\nu\mu}h^\nu{}_b$, $Q_\mu=Q^a{}_{\mu\nu}h^\nu{}_a$, $R=h^{a\mu}R_{a\mu}$.

The sources of the gravitational gauge field

$$T^\mu{}_a=h^{-1}\delta(hL_m)/\delta h^a{}_\mu, \quad J^{\mu b}{}_a=h^{-1}\delta(hL_m)/\delta\Gamma^a{}_{b\mu}$$

are canonical energy-momentum and spin tensors.

When astrophysical and cosmological problems are examined in the Einstein-Cartan theory, the source is naturally taken to be the ideal spinning liquid. In a previous paper [2] we constructed a systematic variational theory of a liquid with spin in a gauge theory of gravitation. The equations of motion of the liquid were obtained in first-order formalism, and it was shown that the sources were

$$T^\mu{}_\nu=-p\delta^\mu{}_\nu+u^\mu(u_\nu(\varepsilon+p)-u^\alpha\overset{*}{\nabla}_\sigma S^\sigma{}_{\alpha\nu}), \tag{3}$$

$$J^\mu{}_{\alpha\beta}=-(1/2)S^\mu{}_{\alpha\beta};\quad S^\mu{}_{\alpha\beta}=u^\mu S_{\alpha\beta}. \tag{4}$$

In the present paper, we consider the application of the model (3)-(4) to cosmological problems in the Einstein-Cartan and more general gauge theories based on Lagrangians that are quadratic functions of twist and curvature. However, before we proceed to the dynamics of the gravitational field proper, and to the study of the effect of spin of matter upon it, we must examine the dynamics of the spin of the liquid (3)-(4) in a gravitational field. As far as we know, this problem has not been discussed in detail in the literature, and some existing publications (see, for example, [2]) contain inaccuracies.

We have to know the dynamics of a spinning liquid in U_4 in order to be able to obtain a closed set of gravitational equations. To achieve this, we must not only specify the sources (3)-(4), but also augment (1)-(2) with the equation of state of the medium and also the translational and rotational equations of motion of the spinning liquid (see [2]). Cosmological models in the Einstein-Cartan theory were previously investigated in [3-5]. From our standpoint, the problem with these publications is that the phenomenological model of the Wayssenhoff liquid was not rigorously employed: instead of the complete equations of spin motion

$$\dot{S}^{\alpha\beta}=u^{\alpha}u_{\mu}\dot{S}^{\mu\beta}-u^{\beta}u_{\mu}\dot{S}^{\mu\alpha} \tag{5}$$

(where the dot represents the substantive derivative, see [2]), they used the so-called spin conservation law

$$\overset{*}{\nabla}_{\mu}(u^{\mu}S_{\alpha\beta})=0. \tag{6}$$

Although all solutions of (6) are also solutions of (5), the reverse is not in general true. This means that the use of (6) instead of (5) to determine spin dynamics in the Einstein-Cartan theory produces an artificial reduction in the size of the manifold of admissible metrics. We shall show that, for a wide class of physically interesting cosmological models, condition (6) can be satisfied only for trivial solutions without spin, $S_{\alpha\beta} = 0$, but the complete equations of motion, given by (5), are always valid. This result follows from the following theorem.

Theorem. The nontrivial spin ($S_{\alpha\beta} \neq 0$) of the Wayssenhoff liquid satisfying the Frenkel condition $u_{\alpha}S^{\alpha\beta}=0$, is conserved in the Einstein-Cartan theory if and only if the liquid moves without acceleration, i.e., $u^{\mu}\nabla_{\mu}u^{\alpha}=0$.

It is well known that to obtain a unique spin dynamics in the phenomenological Wayssenhoff theory we must impose additional conditions on spin density (for example, the Frenkel, Tul'chiev, or Papapetrou conditions). The Frenkel condition

$$u_{\alpha}S^{\alpha\beta}=0, \tag{7}$$

is the most convenient in cosmological problems in the Einstein-Cartan theory, and was employed practically without exception in early works, including [3-5]. In rigorous variational theory [2], we no longer have the freedom to choose the additional conditions, whereas (7) is satisfied automatically.

Before we proceed to a proof, we note that, by virtue of (2), the trace of twist is zero in the Einstein-Cartan theory, so that $\overset{*}{\nabla}_{\mu}=\nabla_{\mu}$ throughout.

The direct statement of the theory is obvious. Thus, suppose that the liquid moves in U_4 without acceleration:

$$u^{\mu}\nabla_{\mu}u^{\alpha}=0. \tag{8}$$

It then follows from (8) and (7) that

$$u_\mu \dot{S}^{\mu\beta} = -(u^\alpha \nabla_\alpha u_\mu) S^{\mu\beta} + u^\alpha \nabla_\alpha (u_\mu S^{\mu\beta}) = 0$$

and, consequently, spin consideration (6) follows from (5).

We must now prove the converse statement. Suppose that the spin is conserved, i.e., we have (6), in which (5) is satisfied identically. The Frenkel condition then has two important consequences.

The algebraic equation (7) means that the 4-velocity of the liquid u^μ is an eigenvector of the matrix $S^{\alpha\beta}$ (spin density) with zero eigenvalue. Differentiating (7) along a current line, i.e., applying to it the operator $u^\mu \nabla_\mu$, and using (6) and (7), we find that

$$(u^\mu \nabla_\mu u_\alpha) S^{\alpha\beta} = 0, \tag{9}$$

i.e., the acceleration vector is also an eigenvector of the spin matrix (again, with zero eigenvalue).

We must now carry out the proof by reductio ad absurdum. We shall suppose that the spin is nontrivial and the acceleration of the liquid is nonzero, so that $\dot{u}^\alpha = u^\mu \nabla_\mu u^\alpha \neq 0$. It is then readily seen that repeated derivatives of (9) along the current lines, $u^\mu \nabla_\mu$, yield new nonzero eigenvectors of the matrix $S^{\alpha\beta}$. We shall show that this (generally infinite) sequence of eigenvectors can be used to select a tetrad which forms an orthonormal basis at each point of U_4.

Let us construct this basis. For the timelike vector $\overset{(0)}{e}_\mu$, $g^{\mu\nu} \overset{(0)}{e}_\mu \overset{(0)}{e}_\nu = 1$, it is natural to take the 4-velocity

$$\overset{(0)}{e}_\mu = u_\mu. \tag{10}$$

Since acceleration is perpendicular to velocity $u^\alpha \dot{u}_\alpha = 0$, it is a spacelike vector. We shall therefore take the spacelike vector of the orthogonal basis in the form

$$\overset{(1)}{e}_\mu = \frac{u^\alpha \nabla_\alpha u_\mu}{\sqrt{-\dot{u}_\beta \dot{u}^\beta}}. \tag{11}$$

Differentiating $\overset{(1)}{e}_\alpha S^{\alpha\beta} = 0$, we obtain a further spacelike vector of the spin matrix, $u^\alpha \nabla_\alpha \overset{(1)}{e}_\mu$. By construction, this is orthogonal to $\overset{(1)}{e}_\mu$, but

$$u^\mu u^\alpha \nabla_\alpha \overset{(1)}{e}_\mu = -\sqrt{-\dot{u}_\alpha \dot{u}^\alpha}. \tag{12}$$

Let $v_\mu = \frac{1}{\sqrt{-\dot{u}_\beta \dot{u}^\beta}} u^\alpha \nabla_\alpha \overset{(1)}{e}_\mu$. We note that this vector differs from zero if and only if $\dot{u}^\alpha \neq 0$. We can now use v_μ to construct the third vector of the orthogonal basis. In fact, the 4-vector

$$\overset{(2)}{e}_\mu = \frac{u_\mu + v_\mu}{\sqrt{1 - v_\alpha v^\alpha}} \tag{13}$$

is spacelike and orthogonal to $\overset{(0)}{e}_\mu$ and $\overset{(1)}{e}_\mu$. By construction, it is also an eigenvector of $S^{\alpha\beta}$.

To obtain the latter vector from the required tetrad, consider the vector $z_\mu = u^\alpha \nabla_\alpha v_\mu$. It is clear that it is also an eigenvector of the spin matrix and satisfies the following orthogonality relations:

$$z^\mu u_\mu = 0, \quad z^\mu \overset{(1)}{e}_\mu = -v_\alpha v^\alpha \sqrt{-\dot{u}_\beta \dot{u}^\beta}. \tag{14}$$

Hence it is clear that z^μ is nonzero if and only if v^μ is nonzero, i.e., when acceleration is nontrivial. We can now readily verify, using (12), (13), and (14) that the vector

$$\overset{(3)}{e}_\mu = c\left(z_\mu - v_\alpha v^\alpha \sqrt{-\dot{u}_\beta \dot{u}^\beta}\,\overset{(1)}{e}_\mu + \overset{(2)}{e}_\mu u^\alpha \nabla_\alpha \left(\sqrt{1 - v_\beta v^\beta}\right)\right) \tag{15}$$

is orthogonal to $\overset{(0)}{e}_\mu, \overset{(1)}{e}_\mu, \overset{(2)}{e}_\mu$, and the constant c can always be chosen so that the normalization condition $\overset{(3)}{e}_\mu \overset{(3)}{e}_\nu g^{\mu\nu} = -1$ is satisfied.

Thus, assuming that $\dot{u}^\alpha \neq 0$, we can use (10), (11), (13), and (15) to define at each point of U_4 a local orthonomal basis $\overset{(a)}{e}_\mu$, $a = 0, 1, 2, 3$. Since, by construction, all these vectors are eigenvectors of the spin matrix

$$\overset{(a)}{e}_\alpha S^{\alpha\beta} = 0,$$

we find from this that any vector field in U_4 is also an eigenvector of the matrix $S^{\alpha\beta}$ with zero eigenvalue. However, this is possible only if $S_{\alpha\beta} = 0$ throughout U_4. Since this conflicts the assumption of nontrivial spin, we must admit that at least one of the vectors $\overset{(i)}{e}_\mu$, $i = 1, 2, 3$ is zero. As already noted, this is possible if and only if there is no acceleration. This proves the theorem.

The above result has important consequences, the most significant of which is that it restricts the class of admissible metrics for the description of cosmological models. Actually, it is readily seen that the requirement of spin conservation (6) imposes constraints both on the Riemann connectedness and on curvature. The former is obvious in comoving coordinates, when the liquid velocity is $u^\mu = \delta^\mu_0$. Zero acceleration is equivalent to $\tilde{\Gamma}^\mu_{00} = 0$ in this case (here and henceforth a tilde denotes Riemann objects constructed from the metric alone: $\tilde{\Gamma}^\alpha{}_{\beta\mu}$ are the Christoffel symbols, $\tilde{\nabla}_\mu$ is the covariant derivative determined by them, $\tilde{R}^\alpha{}_{\beta\mu\nu}$ is the Riemann twist, and so on). The last equation excludes, for example, nonstationary anisotropic cosmological models that arise naturally in the theory (3)-(4). On the other hand (even if we do not consider that the coordinates are comoving), we can use the translational equations of motion to deduce directly a restriction on the curvature.

In point of fact, the translational equations follow from the conservation of the canonical energy-momentum tensor (3) in U_4 [2]:

$$\nabla_\mu T^\mu{}_\nu + 2Q^\alpha{}_{\beta\nu} T^\beta{}_\alpha + (1/2) S^{\alpha\beta} R_{\alpha\beta\mu\nu} u^\mu = 0. \tag{16}$$

where ∇_μ and $R_{\alpha\beta\mu\nu}$ are constructed from the complete Riemann-Cartan connectedness with twist,

$$\Gamma^\alpha{}_{\beta\mu} = \tilde{\Gamma}^\alpha{}_{\beta\mu} + Q^\alpha{}_{\beta\mu} + Q_{\beta\mu}{}^\alpha + Q_{\beta\mu}{}^\alpha. \tag{17}$$

Suppose that spin is conserved in (6). The energy-momentum tensor (3) can then be written in the form

$$T^\mu{}_\nu = -p\delta^\mu_\nu + u^\mu u_\nu (\varepsilon + p). \tag{18}$$

Since by virtue of (7), twist defined by (2) has zero trace in the Einstein-Cartan theory, we find that

$$2Q^{\alpha}{}_{\beta\nu}T^{\beta}{}_{\alpha}=0,\quad \nabla_{\mu}T^{\mu}{}_{\nu}=\tilde{\nabla}_{\mu}T^{\mu}{}_{\nu},\quad u^{\mu}\nabla_{\mu}u^{\nu}=u^{\mu}\tilde{\nabla}_{\mu}u^{\nu}.$$

Transforming the last term in (16) and dividing the Riemann-Cartan curvature into two parts, namely, the Riemann and the twist-dependent part, we find from (2), (4), and (17) that

$$S^{\alpha\beta}R_{\alpha\beta\mu\nu}u^{\mu}=S^{\alpha\beta}\tilde{R}_{\alpha\beta\mu\nu}u^{\mu}+\frac{\varkappa}{4}(\delta^{\mu}_{\nu}-u_{\nu}u^{\mu})\tilde{\nabla}_{\mu}S^{2}+$$
$$+\varkappa(S^{\alpha\beta}S_{\beta\nu}-\delta^{\alpha}_{\nu}S^{2}/2)u^{\mu}\tilde{\nabla}_{\mu}u_{\alpha},\quad S^{2}\equiv S_{\alpha\beta}S^{\alpha\beta}.$$

The above theorem then ensures that the last term is zero and (16) can be rewritten in the form

$$\tilde{\nabla}_{\mu}(-p\delta^{\mu}_{\nu}+u^{\mu}u_{\nu}(\varepsilon+p))+\frac{1}{2}S^{\alpha\beta}\tilde{R}_{\alpha\beta\mu\nu}u^{\mu}+\frac{\varkappa}{8}(\delta^{\mu}_{\nu}-u^{\mu}u_{\nu})\tilde{\nabla}_{\mu}S^{2}=0. \tag{19}$$

Combining u^{ν} with (19), we obtain

$$u^{\mu}\tilde{\nabla}_{\mu}\varepsilon+(\varepsilon+p)\tilde{\nabla}_{\mu}u^{\mu}=0, \tag{20}$$

so that (19) transforms to

$$(-\delta^{\mu}_{\nu}+u^{\mu}u_{\nu})\tilde{\nabla}_{\mu}\left(p-\frac{\varkappa}{8}S^{2}\right)+\frac{1}{2}S^{\alpha\beta}\tilde{R}_{\alpha\beta\mu\nu}u^{\mu}=0, \tag{21}$$

where we have again used the fact that the acceleration is zero.

We thus find that, because spin is conserved (6), the Mathisson force given by the last term in (21) must be potential to ensure that the pressure force determined by the gradient of the effective pressure $p_{eff}=p-(\varkappa/8)S^{2}$ is balanced. Since the Mathisson force is constructed directly from the Riemann curvature, the condition that the force must be potential produces a significant restriction on the possible structure of $\tilde{R}^{\alpha}{}_{\beta\mu\nu}$.

We conclude with two technical points. The internal consistency of the theory can be checked by transforming to the so-called effective form of the Einstein-Cartan equations: by substituting the twist (2), (4) in (1) and by isolating in the Einstein tensor the Riemann and non-Riemann parts, we ensure, as is well known, the appearance on the right side of (1) of the hydrodynamic energy-momentum tensor with $\varepsilon_{eff}=\varepsilon-(\varkappa/8)S^{2}$, $p_{eff}=p-(\varkappa/8)S^{2}$. It is then readily verified that the direct evaluation of the covariant divergence of the effective equations (1) leads to (20), (21), subject to the condition that the spin is conserved. The second point is concerned with (7) which, as we have seen, plays a significant part in the proof of the theorem. In contrast to rigorous variational theory [2], the Frenkel condition in the phenomenological model of the Wayssenhoff liquid is not the only possible one. However, it is clear that an analogous theorem can be proved for the Tul'chiev or Papapetrou conditions, which again leads to restrictions (although somewhat different) on the metric, the Christoffel symbols, and the Riemann curvature.

Thus, the above analysis of spin dynamics in the Einstein-Cartan theory leads to the following conclusion: the use of spin conservation in place of the complete equations of spin motion is in general incorrect. Specific cosmological models in which spin (4) is not conserved will be examined elsewhere.

REFERENCES

1. A. Trautman, Sympos. Math., vol. 12, p. 139, 1973; F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester, Rev. Mod. Phys., vol. 48, pp. 393-

416, 1976; V. N. Ponomarev, A. O. Barvinskii, and Yu. N. Obukhov, Geometrodynamic Methods and the Gauge Approach to the Theory of Gravitational Interactions [in Russian], p. 168, Moscow, 1985; D. D. Ivanenko, P. I. Pronin, and P. I. Sardanashvili, Gauge Theory of Gravitation [in Russian], p. 136, Moscow, 1985.

2. V. A. Korotkin and Yu. N. Obukhov, Vestn. Mosk. Univ. Fiz. Astron. Ser. 3 [Moscow University Physics Bulletin], no. 4, pp. 6-11, 1987.

3. B. Kuchowicz, Preprint N-51, Inst. Astr. Pol. Acad. Sci., 1975; J. Phys. A, vol. 8, pp. L29-33, 1975.

4. A. Trautman, Nature, vol. 242, p. 7, 1973; W. Kopczinski, Phys. Lett. A, vol. 39, pp. 219-220, 1972; Ibid., A, vol. 43, pp. 63-64.

5. I. S. Nurgaliev and V. N. Ponomarev, Izv. Vyssh. Uchebn. Zaved. Fiz., no. 10, pp. 63-65, 1982; I. S. Nurgaliev and V. N. Ponomarev, Ibid., no. 9, pp. 32-35, 1982

4 June 1986

Chair of Theoretical Physics