EXACT DYNAMIC THEORIES OPERATING ON A GIVEN BACKGROUND

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Exact dynamic theories induced by the subdivision of the variables in the original field theory into a background and a finite dynamic part are constructed. They exhibit a certain nonequivalence under different cases of the subidivied variables. Gauge invariance that follows from displacements in 4-space and dynamic theories in GTR are discussed.

The subdivision of the field variables in some initial field theory into a background and a dynamic part has a long history. In gravitation, it has frequently been used since the publication of [1] to solve classical relativistic problems and to quantize the weak gravitational field. We are not able to reproduce here the extensive bibliography on this subject; a partial listing can be found in [2].

The principle used to construct a dynamic theory, i.e., a theory of a dynamic field on a given background of the same nature, is valid for any theory describing a single field or several fields, and presented in either second or first order formalisms. However, dynamic theories are probably most effective in gravitation because the background gravitational field can be regarded as 4-space that serves as the arena for the action of dynamic gravitational fields. This means that such fields can be discussed in the same way as nongravitational fields, e.g., on a background of the Schwarzschild solution or cosmologic models [3]. Dynamic theories are necessary in the quantization of the gravitational field in the semiclassical approximation. Effective action and the conformal anomaly are discussed in [4].

We shall not assume that the dynamic field is small and will construct a dynamic theory exactly and independently of whether or not it can be specified in a closed form (see [5-7] in relation to gravitation) or only in the form of the sum of an infinite power series in terms of the dynamic variables. This type of construction is reasonable when it is possible to isolate some smooth or slowly varying part of the field as the background, and when the dynamic theory describes the entire range of phenomena covered by the original theory with the same degree of completeness. For example, the closed Friedmann universe is satisfactorily described by the dynamic theory [8].

In this paper, we shall introduce a series of statements in relation to dynamic theories generally, and will illustrate them by the theory of the gravitational field or a system of interacting gravitational and matter fields [7].

In section 1, we consider the properties of an arbitrary theory (not necessarily dynamic) in relation to the redefinition of the field variables. The transformed theory will be found to be equivalent to the original theory.

In section 2, we determine the dynamic Lagrangian and obtain the dynamic equations. We also establish the "background current" which corresponds to the variation of the dynamic Lagrangian with respect to the background variables. Dynamic theories obtained for different cases of the field variables turn out to be nonequivalent in a certain sense. In gravitation on a flat 4-background, the energy-momentum tensors differ by the divergence of the "superpotential" (in the traditional terminology). The superpotential is mentioned in [9] in the quadratic approximation. We shall show that the exact theory contains terms that lead to the superpotential.

In section 3, we investigate the invariance of dynamic theories of gravitational and matter fields and their gauge transformations induced by finite displacements of 4-space.

1. Redefinition of fields. Consider a set of fields f^A and P^{α} where the indices A and α correspond to arbitrary and generally different transformation properties. Suppose the field P is an auxiliary, "external" field relative to f. An example of this situation is provided by an arbitrary matter field imbedded in an external metric.

Thus, consider the following action functional

$$S=\int d^4x L(f|P). \qquad (1)$$

By varying (1), we obtain the equations of motion for the field f and the "external current":

$$\frac{\delta L}{\delta f^A}=0, \qquad (2)$$

$$\tau^f_{\alpha}=\frac{\delta L}{\delta P^{\alpha}}. \qquad (3)$$

When the field P is a metric tensor, the external current (3) is proportional to the energy-momentum tensor.

We now introduce the following transformation of variables:

$$f^A=f^A(\tilde{f}, P) \qquad (4)$$

and assume that this transformation has an inverse and that it does not contain the derivatives of the fields. The transformation from variables with a particular position of the indeces to variables with a different position of the indices is an example of this.

The equations of motion and the external current corresponding to the field $\tilde{f}$ are related to the equations of motion and the external current for the field f by

$$\frac{\delta\tilde{L}}{\delta\tilde{f}^A}=\frac{\partial f^B}{\partial\tilde{f}^A}\frac{\delta L}{\delta f^B}\bigg|_{f=f(\tilde{f},P)}, \qquad (5)$$

$$\tilde{\tau}^{\tilde{f}}_{\alpha}=\left[\tau^f_{\alpha}+\frac{\partial f^A}{\partial P^{\alpha}}\frac{\delta L}{\delta f^A}\right]_{f=f(\tilde{f},P)}. \qquad (6)$$

It is clear that the field theories f, P and $\tilde{f}$, P are equivalent in the sense that the equations of motion (2) and (5) follow from one another and the external currents (3) and (6) are identical when the equations of motion are the same.

These propositions are equally valid under a change of variables in dynamic heories.

2. **Dynamic theories.** Let us first construct a "dynamic theory." Suppose hat a field or a set of fields denoted by $\mathcal{Q}^A$ can be described by the action

$$S=\int d^4x L(\mathcal{Q}). \tag{7}$$

et us specify the variables $\mathcal{Q}$ in some particular way (by the position of inices, the weight of tensorial densities, and so on). Next, let us subdivided $\mathcal{Q}^A$ into the background (Q^A) and dynamic (q^A) parts in the sense of the followng exact equation:

$$\mathcal{Q}^A=Q^A+q^A. \tag{8}$$

e now define exactly, without successive approximations, the dynamic Lagrangian f the field q:

$$L^{\text{dyn}}(q|Q)=L(Q+q)-L^0(Q)-L^1(q|Q), \tag{9}$$

here

$$L^0(Q)=L(Q), \tag{10}$$

$$L^1=q^A\frac{\delta L^0}{\delta Q^A}. \tag{11}$$

he background quantities now play the same part as the field P in section 1.

To within the divergence, the original Lagrangian L(Q + q) can be repreented by a generalized Taylor series involving Lagrange derivatives with respect o the background quantities (see, forexample, [10]). Actually, (10) and (11) re the zero and first order terms of this series, respectively.

The background equations of motion follow from (10):

$$\frac{\delta L^0}{\delta Q^A}=0. \tag{12}$$

e note that the coefficient of q^A in (11) is the left hand side of (12), but he background equations of motion cannot be taken into account in L^{dyn} prior o the variation.

By varying (9) with respect to q and Q, we obtain the dynamic equations of otion and the "background current" analogous to the "background current" in 3). We now use the obvious property $\delta L/\delta Q=\delta L/\delta q$ of the above subdivision, and rite the equations of motion and the background current in the form

$$\frac{\delta L^{\text{dyn}}}{\delta q^A}=\frac{\delta}{\delta Q^A}\left[L(Q+q)-L^0(Q)\right]=0, \tag{13}$$

$$\tau^q_A=\frac{\delta L^{\text{dyn}}}{\delta Q^A}=\frac{\delta L^{\text{dyn}}}{\delta q^A}-\frac{\delta}{\delta Q^A}q^B\frac{\delta L^0}{\delta Q^B}. \tag{14}$$

e shall assume from now on that the Lagrange derivative acts from the right on he entire expression.

It is clear from (13) and (14) that the background current is the "source" f the linear part of the equations of motion, which can be rewritten in the form

$$\frac{\delta}{\delta Q^A}q^B\frac{\delta}{\delta Q^B}L^0=-\tau^q_A. \tag{15}$$

Once we have the dynamic theory, we can always construct an approximate theory of a given order. The first term of the expansion in L^{dyn} is the quadratic term in the expansion of the original Lagrangian:

$$L^2 = \frac{1}{2} q^A \frac{\delta}{\delta Q^A} q^B \frac{\delta}{\delta Q^B} L^0. \qquad (16)$$

Since the Lagrange derivatives commute, we can vary (16) to obtain the linear equations of motion for the field q and the background current that is quadratic in q:

$$\frac{\delta}{\delta q^A} L^2 = \frac{\delta}{\delta Q^A} q^B \frac{\delta}{\delta Q^B} L^0 = 0, \qquad (17)$$

$$\tau^q_A = \frac{\delta}{\delta Q^A} L^2 = \frac{1}{2} \frac{\delta}{\delta Q^A} q^B \frac{\delta}{\delta Q^B} q^C \frac{\delta}{\delta Q^C} L^0. \qquad (18)$$

These two equations can also be obtained in a different way, i.e., as, respectively, the linear and quadratic parts of the expansion of the equations of motion that follow directly from the original Lagrangian (7).

We shall now establish the differences that arise in the dynamic Lagrangians and the corresponding equations of motion and background currents when two dynamic series are constructed for a different case of the independent variables in the action (7). Suppose that

$$Q^A_1 = Q^A_1 + q^A_1, \qquad (19)$$

$$Q^A_2 = Q^A_2 + q^A_2, \qquad (20)$$

where, of course, $Q^A_2 = Q^A_2(Q^B_1)$. We now write out the dynamic Lagrangians corresponding to the subdivisions (19) and (20):

$$L^{dyn}_1 = L(Q^A_1 + q^A_1) - L^0(Q^A_1) - q^A_1 \frac{\delta}{\delta Q^A_1} L^0, \qquad (21)$$

$$L^{dyn}_2 = L(Q^A_2 + q^A_2) - L^0(Q^A_2) - q^A_2 \frac{\delta}{\delta Q^A_2} L^0. \qquad (22)$$

To compare (21) and (22), we express the former in terms of the dynamic variables q_1. We represent q_2 by a series in terms of q_1, and recall that $Q_2 = Q_2(Q_1)$:

$$\left.\begin{aligned} q^A_2 &= \frac{\partial Q^A_2}{\partial Q^B_1} q^B_1 + \alpha^A(q_1|Q_1), \\ \alpha^A &= \frac{1}{2!} \frac{\partial^2 Q^A_2}{\partial Q^B_1 \partial Q^C_1} q^B_1 q^C_1 + \frac{1}{3!} \frac{\partial^3 Q^A_2}{\partial Q^B_1 \partial Q^C_1 \partial Q^D_1} q^B_1 q^C_1 q^D_1 + \dots \end{aligned}\right\} \qquad (23)$$

Substituting (23) in (22), and using the equation $\frac{\partial Q^B_2}{\partial Q^A_1} \frac{\delta}{\delta Q^B_2} = \frac{\delta}{\delta Q^A_1}$, we obtain the difference between (21) and (22):

$$L^{dyn}_1 (q_1|Q_1) - L^{dyn}_2 (q_2(q_1|Q_1)|Q_2(Q_1)) = \beta^A(q_1|Q_1) \frac{\delta}{\delta Q^A_1} L^0, \qquad (24)$$

where

$$\beta^A = (\partial Q^A{}_1/\partial Q^B{}_2) \cdot \alpha^B.$$

The equations of motion that follow from $L_1{}^{dyn}$ and $L_2{}^{dyn}$ will differ by the following quantity when expressed in terms of the variables q_1:

$$\frac{\delta}{\delta q_1^A} L_1^{\text{dyn}} - \frac{\partial q_2^B}{\partial q_1^A} \frac{\delta}{\delta q_2^B} L_2^{\text{dyn}} = \frac{\partial \beta^B}{\partial q_1^A} \frac{\delta}{\delta Q_1^B} L^0.$$

Thus, the differences between the two subdivisions (19) and (20) in L^{dyn} and the equations of motion of the field q are proportional to the background equations of motion (12) and vanish when the latter are satisfied.

We shall show that the difference (24) provides a contribution to the background current that does not vanish in the background equations of motion (12) or the equations of motion (13). Let us take the Lagrange derivative of (24) with respect to Q_1, so that we obtain

$$\tau_{1A} - \frac{\partial Q_2^B}{\partial Q_1^A} \tau_{2B} = \frac{\partial q_2^B}{\partial Q_1^A} \frac{\delta}{\delta q_2^B} L_2^{\text{dyn}} + \frac{\partial \beta^B}{\partial Q_1^A} \frac{\delta}{\delta Q_1^B} L^0 + \frac{\delta}{\delta Q_1^A} \beta^B \frac{\delta}{\delta Q_1^B} L^0. \qquad (25)$$

The first and second terms on the right hand side of (25) are respectively proportional to the equations of motion and the background equations of motion, whereas the third term, in which β is independent of Q_1, is the linear part of the operator in the equations of motion, applied to β.

In the case of "pure gravitation," described by the Hilbert action $-(1/2)\sqrt{-g}R$, the dynamic variables are often taken to be, for example, $g_{\mu\nu}$, $g^{\mu\nu}$, $\sqrt{-g}g^{\mu\nu}$, and so on. In the corresponding dynamic theories, the energy-momentum tensors are proportional to the background current. The difference between the energy-momentum tensors of two dynamic theories, that is due to the third term in (25), is the double covariant divergence or, in the case of the flat 4-background, the divergence of the 3-index quantity called the superpotential.

We emphasize that the emergence of the terms (25) is due precisely to the different choice of the subdivided variables and the resulting difference between the dynamic Lagrangians, which cannot be due to the simple redefinition of the variables $q_2 = q_2(q_1|Q_1)$ described in section 1.

3. Gauge invariance. Consider a covariant theory of "gravitation" + "matter," described by the single variable $\boldsymbol{Q}$. We shall show that this theory and the corresponding dynamic theory exhibit a specific form of gauge invariance.

Let us transform the coordinates as follows:

$$x'^\alpha = f^\alpha(x) = \left[\exp\left(\xi^\beta(x)\frac{\partial}{\partial x^\beta}\right)\right] x^\alpha, \qquad (26)$$

where "exp" is regarded as a differential operator and ξ is a vector that, in general, is not small.

The standard operation used to evaluate the Lie derivatives of geometric objects consists of the transformation (26) of the coordinate system, followed by return to the point having the value x in the new system x'. When this operation is performed exactly (and not in the first order in ξ), it can be interpreted as a finite displacement of space in the direction of the vector ξ. The transformation of any geometric object Ω then takes the form

$$\Omega'(x) = (\exp \mathcal{L}_\xi)\Omega(x) = \Omega(x) + \mathcal{L}_\xi\Omega + \frac{1}{2!}\mathcal{L}_\xi(\mathcal{L}_\xi\Omega) + \ldots, \qquad (27)$$

where $\mathcal{L}_\xi$ is the usual Lie differential [11].

Thus, transformations such as (26) generate transformations of the variables such as (27) in theories with action (1):

$$f' = (\exp \mathscr{L}_\xi) f, \quad P' = (\exp \mathscr{L}_\xi) P. \tag{28}$$

The action given by (1) is invariant under (28) because

$$L(f'|P') = L(\exp \mathscr{L}_\xi f | \exp \mathscr{L}_\xi P) = \exp \mathscr{L}_\xi L(f|P) = L(f|P) - (\xi^\alpha L)_{,\alpha} + \\ + \frac{1}{2!} (\xi^\beta (\xi^\alpha L)_{,\alpha})_{,\beta} - \ldots, \tag{29}$$

i.e., the terms in the infinite series, after the second term, form the complete divergence. Here and henceforth we use the fact that the operator $\exp \mathscr{L}_\xi$ commutes with the function symbol, which can readily be verified.

The equations of motion and the background currents are "gauge covariant," i.e., they transform into themselves:

$$\frac{\delta L}{\delta f'} = \exp \mathscr{L}_\xi \frac{\delta L}{\delta f}, \quad \frac{\delta L}{\delta P'} = \exp \mathscr{L}_\xi \frac{\delta L}{\delta P}.$$

In the dynamic theory with the subdivision defined by (8), the gauge transformations will be applied to the dynamic variables. Let

$$q' = q + (\exp \mathscr{L}_\xi - 1)(Q + q). \tag{30}$$

This means that we have the following correspondence with the original theory: the quantities $\mathcal{Q} = Q + q$ and $\mathcal{Q}' = Q + q'$ are related by the transformation (28).

We shall now examine how our set of dynamic quantities transforms under (30). For $L^{\rm dyn}$, we have

$$L^{\rm dyn}(q') = L^{\rm dyn}(q) + (\exp \mathscr{L}_\xi - 1) L(Q + q) - \\ - [(\exp \mathscr{L}_\xi - 1)(Q^A + q^A)] \frac{\delta}{\delta Q^A} L^0(Q), \tag{31}$$

where the second term is the divergence (compare this with (29)) and the third term vanishes when the background equations of motion are satisfied. Thus, the dynamic action is gauge invariant.

After substituting (30), we find that the dynamic equations of motion

$$\frac{\delta L^{\rm dyn}}{\delta q'^A} = \exp \mathscr{L}_\xi \frac{\delta L^{\rm dyn}}{\delta q^A} + (\exp \mathscr{L}_\xi - 1) \frac{\delta L^0}{\delta Q^A}, \tag{32}$$

become obviously "gauge covariant" when the background equations of motion are satisfied.

The background current is found to be

$$\tau_A(q') = \tau_A(q) + (\exp \mathscr{L}_\xi - 1) \frac{\delta L^0}{\delta Q^A} + (\exp \mathscr{L}_\xi - 1) \frac{\delta L^{\rm dyn}}{\delta q^A} - \\ - \frac{\delta}{\delta Q^A} [(\exp \mathscr{L}_\xi - 1)(Q^B + q^B)] \frac{\delta}{\delta Q^B} L^0. \tag{33}$$

Because of its structure (see (15)), the background current is not gauge invariant even when the background equations of motion and the dynamic equations of motion that appear in the second and third terms of (33) are satisfied. The fourth term in (33) provides a contribution that does not vanish on the equations of motion or the background equations of motion.

At first sight it would appear that (32), (33) and (5), (6) are in conflict. We recall that the latter relate the equations of motion and background current obtained directly by the transformation of their variables to the equations of motion and background current derived from the transformed Lagrangian. Here we directly transform only the equations of motion and the background current themselves. On the other hand, if we determine these quantities from the transformed L^{dyn} (31), we find that they correspond exactly to (5), (6). We emphasize that, in (32) and (33), we are dealing with form-invariance, i.e., invariance of the form of the function of our dynamic variables.

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