DIFFRACTION OF ACOUSTIC WAVES BY A HALF-PLANE PARALLEL TO THE FREE SURFACE OF A STRATIFIED COMPRESSIBLE LIQUID

R. R. Gabdullin and A. K. Shatov

Vestnik Moskovskogo Universiteta. Fizika,
Vol. 42, No. 6, pp. 46-52, 1987

UDC 517.958:535.4

An analysis is given of the diffraction of acoustic waves by a rigid-half plane parallel to the free surface of a stratified compressible liquid. The solution is given in the form of an integral whose asymptotic behavior in the far-field zone is examined.

We investigate the scattering of plane waves by the edge of a half-plane in a stratified compressible liquid, when the plane is parallel to the free surface of the liquid.

By a stratified liquid we mean a liquid whose steady-state characteristics (in the present context, the density) vary only along a chosen particular direction. Studies of the dynamics of waves in different inhomogeneous (including stratified) liquids are of interest in connection with problems in geophysics, oceanology, atmospheric physics, applications of cryogenic liquids in technology, pollution of the environment, and a number of other questions [1].

1. Let us consider the two-dimensional motion of a stratified liquid in a uniform gravitational field relative to a Cartesian frame xOz in which the Oz axis points against the constant gravitational acceleration $g = (0, -g)$.

Small oscillations of the liquid relative to the stationary state, which we shall assume to be adiabatic and have the time dependence $\exp(-i\omega t)$, will be described by the following set of equations:

$$\begin{gathered} -i\omega\rho_0\mathbf{v} + \nabla p = \rho\mathbf{g}; \quad -i\omega\rho + (\nabla\rho_0, \mathbf{v}) + \rho_0 \operatorname{div}\mathbf{v} = 0 \\ -i\omega\rho + (\nabla\rho_0, \mathbf{v}) = \frac{1}{c^2}(-i\omega p + (\nabla p_0, \mathbf{v})) \end{gathered} \tag{1a}$$

where $\mathbf{v}=(v_1, v_2)$, ρ, and p are the amplitudes of the particle velocity vector, density perturbation, and dynamic pressure, respectively, and $\rho_0(z)$ and $p_0(z)$ are the stationary distributions of density and pressure, related by $\nabla p_0=\rho_0\mathbf{g}$. The first two equations in (1a) describe the conservation of the momentum of the particles of the liquid, whereas the third equation is the equation of continuity. The last equation is the equation of state. In accordance with the above definition of a stratified liquid, we put $\rho_0(z)=\rho_0(0)e^{-2\beta z}$, $\beta>0$, and consider that β and c (the adiabatic velocity of sound) are constants in the approximation that we have adopted.

If we substitute $\Psi=pe^{\beta z}$, we can reduce (1a) to the single equation [2]

$$\frac{1}{a^2}\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial z^2}+k^2\Psi=0, \qquad (1)$$

where $a^2=\omega^2/(\omega^2-\omega_0^2)$, $k^2=\omega^2/c^2-\beta^2$, and the quantities p, ρ, and v are related to Ψ by

$$p=e^{-\beta z}\Psi;\quad \rho=\frac{\omega_0^2}{\mu_0-\beta}e^{-\beta z}(1-a^2)\left(\frac{d}{\partial z}-\frac{a^2\mu_0+\beta}{1-a^2}\right)\Psi$$
$$v_1=\frac{e^{\beta z}}{i\omega}\frac{\partial\Psi}{\partial x};\quad v_2=\frac{a^2}{i\omega}e^{\beta z}\left(\frac{\partial}{\partial z}+\mu_0\right)\Psi \qquad (2)$$

where $\mu_0=\beta-\omega_0^2/g$ and $\omega_0^2=2\beta g-g^2/c^2$. Stratification is said to be stable when $\omega_0^2>0$, in which case $|\mu_0|<\beta$.

We note that we have chosen the case of acoustic oscillations, i.e., $a^2>0$, $k^2>0$, and will not consider internal waves for which $a^2<0$, $k^2<0$ for which plane-wave solutions can also be found [2].

2. Suppose the liquid occupies the half-space $z<h$ and contains, parallel to its free surface $z=h$ at a depth $h>0$, a rigid half-plane $\Gamma\equiv\{z=0,\ x<0\}$. The wave $I=\{0<z<1,\ x<0\}$ is incident on the edge of Γ from the region

$$\Psi_0=e^{ik_0x}\left(\operatorname{ch}\gamma_0z-\mu_0\frac{\operatorname{sh}\gamma_0z}{\gamma_0}\right),$$

where $k_0=a\sqrt{k^2+\gamma_0^2}$, and γ_0 is the minimum positive real root of the equation

$$\gamma\operatorname{ctg}\gamma h=\frac{\mu\mu_0-\gamma^2}{\mu-\mu_0},\quad \mu=\beta-\omega^2/g. \qquad (3)$$

On the free surface $z=h$ we have the dynamic condition $p(x,h)=\zeta(x)\rho_0(h)$, where $\zeta(x)=(-1/i\omega)v_2(x,h)$ is the displacement above the free surface. In terms of the function $\Psi(x,z)$, this condition takes the form $(\partial/\partial z+\mu)\Psi|_{z=h}=0$ (see (2)). Since the half-plane Γ is impermeable, the function Ψ must satisfy the condition $(\partial/\partial z+\mu_0)\Psi|_\Gamma=0$.

If in the region $0<z<h$, the resultant wave field Ψ_Σ is written in the form $\Psi_\Sigma=\Psi_0+\Psi$, whereas in the region $z<0$, the resultant field is denoted by Ψ, then the problem of diffraction of Ψ_0 by the edge of Γ can be formulated as follows: we seek the function $\Psi(x,z)$ defined in $\{z<h\}\setminus\Gamma$ that satisfies equation (1) for $z\neq0$, subject to the above boundary conditions and the following consequences of the requirement of continuity of dynamic pressure and the component v_2 for $z=0$, $x>0$:

$$\Psi(x,+0)-\Psi(x,-0)=-e^{ik_0x}$$

and

$$\frac{\partial\Psi}{\partial z}(x,+0)-\frac{\partial\Psi}{\partial z}(x,-0)=\mu_0e^{ik_0x}.$$

On the edge (0, 0), we impose the condition $\frac{d\Psi}{dz}(x,0\pm0)=0(x^{-1/2})$, $x\to0\pm0$, which follows from the absence of additional sources on the edge of Γ. This is equivalent to saying that the total energy flux through any surface surrounding the edge must be zero.

The conditions at infinity will be formulated by demanding that waves produced as a result of diffraction transport energy to infinity.

3. The above problem will now be solved by the Wiener-Hopf method in the Jones interpretation. A detailed description of this can be found in [3].

The solution has the form

$$\Psi(x, z) = \int_{-\infty}^{\infty} \Phi_1(\alpha, z) e^{-\gamma|z-2h|-i\alpha x} d\alpha + \int_{-\infty}^{\infty} \Phi_2(\alpha, z) e^{-\gamma|z|-i\alpha x} d\alpha, \qquad (4)$$

where $\gamma = \gamma(\alpha) = (1/a)\sqrt{\alpha^2 - k^2a^2}$, with the chosen branch $\gamma(0) = -ik$, $\gamma(\alpha) \to \infty$ as $|\alpha| \to \infty$ on the real axis in the plane of α. The cut used to define this branch joins the branch points $\pm ak$ via an infinitely distant point along the rays $[ak+i0,\ ak+i\infty)$ and $[-ak-i0,\ -ak-i\infty)$. The integrals in (4) are evaluated along the real axis, bypassing the positive singularities below and the negative singularities above. In (4),

$$\Phi_1(\alpha, z) = \Phi(\alpha) R(\alpha), \quad \Phi_2(\alpha, z) = \Phi(\alpha)\left(1 - \frac{2\mu_0}{M_0(\alpha)} \chi(z)\right),$$

$$\Phi(\alpha) = -\frac{i}{2\pi} L_-(-k_0) \frac{L_+(\alpha) M(\alpha) e^{i\gamma h}}{2\gamma G(\alpha)\cdot(\alpha + k_0)},$$

where

$$R(\alpha) = \frac{\gamma - \mu}{\gamma + \mu}, \quad M(\alpha) = \gamma + \mu, \quad M_0(\alpha) = \gamma + \mu_0,$$

$$G(\alpha) = (\mu_0 - \mu)\operatorname{ch}\gamma h + (\mu\mu_0 - \gamma^2)\frac{\operatorname{sh}\gamma h}{\gamma}, \quad \chi(z) = \begin{cases} 0, & z < 0, \\ 1, & z > 0. \end{cases}$$

The functions $L_\pm(\alpha)$ are obtained by factorizing the function $L(\alpha) = \frac{M_0(\alpha) G(\alpha)}{M(\alpha) e^{\gamma h}}$ into the product $L_+(\alpha)L_-(\alpha)$ where $L_+(\alpha)$ and $L_-(\alpha)$ are analytic and nonzero in the upper and lower half planes, respectively. The factorization is carried out on the basis of the factorization of functions such as $e^{\gamma h}$, $M(\alpha)$, $G(\alpha)$ of [3,4,5], respectively.

The final result is

$$L_-(\alpha) = L_+(-\alpha), \quad L_+(\alpha) = \frac{M_{0+}(\alpha)}{M_+(\alpha)} \exp\left\{\frac{i\gamma h}{\pi} \ln\frac{\alpha + \gamma a}{ka} - \right.$$

$$\left. - \frac{i\alpha h}{\pi a}\left(1 - C_0 - \ln\frac{ikh}{2\pi}\right)\right\} \sqrt{G(0)}\left(1 + \frac{\alpha}{\alpha_0}\right) \prod_{m=1}^{\infty}\left(1 + \frac{\alpha}{\alpha_m}\right) e^{-\frac{i\alpha h}{\pi m a}}, \qquad (5)$$

where we have taken the principal branch of the function $\ln\alpha = \ln|\alpha| + i\arg\alpha$, $-\pi/2 < \arg\alpha < 3\pi/2$, C_0 is the Euler constant, $G(0) = \operatorname{ch} kh - \operatorname{sh} kh$, and α_m, $m = 0, 1, 2 \ldots$ are the zeros of the function $G(\alpha)$ lying in the range $\{0 < \arg\alpha < \pi\}$ on the plane of the complex variable α.

The factorizations $M(\alpha) = M_+(\alpha)M_-(\alpha)$, $M_0(\alpha) = M_{0+}(\alpha)M_{0-}(\alpha)$ are performed by analogy with [3], and we merely note that the form of the factorization depends on the signs of μ and μ_0.

The function $G(\alpha)$ has only simple zeros $\pm\alpha_m$ which form a denumerable set without limit points, with the exception of an infinitely distant point. The zeros of $G(\alpha)$ are found after the solution of (3) has been carried out in the complex plane: for each pair of roots of (3) there is a pair of zeros of $G(\alpha)$, $\pm\alpha_m$, that are related with γ_m by $a^2(\gamma_m^2+k^2)=\alpha_m^2$. The roots $+\alpha_m$ are real and positive for m = 0, 1, ..., n and purely imaginary with positive imaginary part for m = n +1, n +2, ... If we number them in order of increasing modulus, we obtain the following asymptotic estimate that is valid for $m \gg n + 1$:

$$\alpha_m = i\left\{\frac{\pi m a}{h} - \frac{a}{\pi m}\left(\mu_0 - \mu + \frac{k^2 h}{2}\right) + O\left(\frac{1}{m^3}\right)\right\}. \qquad (6)$$

The number n depends on ω and the parameters β, c, and is found from the relation $y_n^2 < \omega^2/c^2 - \beta^2 < y_{n+1}^2$, where y_n, n = 1, 2, ... are the positive roots of $y \operatorname{ctg} y = -\beta$, numbered in order of their increasing value. When $\omega^2/c^2 - \beta^2 < y_1^2$, it is assumed that n = 0.

4. We now turn to the analysis of the field described by (4). It will be convenient to divide z < h into 2 regions, namely, region I (see section 2) and $\mathrm{II} = \{z < h\} \setminus I$.

Region I may be referred to as the waveguide region formed by the solid wall Γ and the free surface of the liquid. When the wave field is analyzed in region I, we write (4) in the form of a single integral and close the contour of integration in the upper half-plane. The only poles of the resulting integrand in the interior of the contour of integration are $+\alpha_m$, m = 0, 1, 2, ... Applying the Cauchy theorem to the resulting integral, we can write the expression for Ψ in the form of a sum over the residues of the integrand, in which case

$$\Psi_z = \Psi_0 + \sum_{m=0}^{\infty} A(\alpha_m)\left(\operatorname{ch}\gamma_m(h-z) + \mu\frac{\operatorname{sh}\gamma_m(h-z)}{\gamma_m}\right)e^{-i\alpha_m x}, \qquad (7)$$

where $A(\alpha) = \dfrac{L_+(k_0)\,L_+(\alpha)}{G'(\alpha)(\alpha+k_0)}$.

We note that all the terms under the summation sum have the significance of the eigenwaves of the waveguide. The first n + 1 of them (m = 0, 1, ..., n) are harmonic functions of x whereas all the others (m = n + 1, n + 2, ...) decay exponentially with decreasing x < 0. We note that $n + 1 \geq 1$, i.e., there is at least one propagating wave in x.

All the terms in the sum in (7) are harmonic functions of z, except for the first two (m = 0, 1) for $-\mu\mu_0 h/(\mu_0-\mu) < 1$ and the first (m = 0) for $-\mu\mu_0 h/(\mu_0-\mu) > 1$, since γ_m, m = 2, 3, ... in the first case, or m = 1, 2, ... in the second, are purely imaginary numbers.

The inequality $-\mu\mu_0 h/(\mu_0-\mu) > 1$ is satisfied when $\mu_0 h < 1$ for $\omega > \beta c$ and when $1 + gh/c^2 > \mu_0 h > 1$ for $\omega < \omega_s$. The inequality $-\mu_0\mu h/(\mu_0-\mu) < 1$ is satisfied when $1 < \mu_0 h < 1 + gh/c^2$ for $\omega > \omega_s$ and $\mu_0 h > 1 + gh/c^2$ for $\omega > \beta c$. In these expressions ω_s is the characteristic frequency of the waveguide, given by:

$$\omega_s^2 = [\beta^2 gh - (1+\beta h)\omega_0^2]/(\mu_0 h - 1).$$

The series in (7) is absolutely convergent, together with all its derivatives, for all $z \in (0, h)$ and x < 0.

When the field in region II is analyzed, the first integral is conveniently taken in terms of polar coordinates (r_1, θ_1) with the center at $O_1 = (0, 0)$, and the second integral in terms of (r_2, θ_2) with the center at $O_2 = (0, 2h)$. The angles θ_1 and θ_2 are measured, respectively, from the Ox and $\{z=2h, x>0\}$ axes in the anticlockwise direction.

Closing the contours of integration in a suitable way, taking into account the residues of the integrands at the points $k_0, \varkappa_0 = a\sqrt{k^2+\mu_0^2} = \omega/c$, $\varkappa = a\sqrt{k^2+\mu^2} = \omega^2/g$, and carrying out the asymptotic estimate of the integrals over contours running around cuts, we obtain the following expression for the resultant field in II:

$$\Psi_{\Sigma} = \chi(-\mu_0)\Psi_k + \chi(-\mu)\Psi_s + \Psi_1^d + \Psi_2^d, \tag{8}$$

where $\Psi_k = \frac{a^2\mu_0}{\varkappa_0(k_0+\varkappa_0)} L_+(k_0) L_+(\varkappa_0) e^{-|\mu_0 z| - i\varkappa_0 x}$ is the Kelvin wave under the half-plane Γ, which is a progressive way propagating in the negative x direction with velocity c, and

$$\Psi_s = -\frac{a^2\mu(\mu_0-\mu)}{\varkappa(k_0-\varkappa)} L_+(k_0) L_+(\varkappa) e^{-|\mu z| + i\varkappa x}$$

is a surface wave propagating over the free surface in the positive direction of x with the phase velocity g/ω.

The following asymptotic expressions are valid for the terms $\Psi_j{}^d$, $j = 1, 2$ as $k, r_j \to \infty$:

$$\Psi_j^d = \Phi(\alpha(\theta_j), \quad r_j \sin\theta_j) \frac{ka|\sin\theta_j|}{\varphi^{3/2}(\theta_j)} \sqrt{\frac{2\pi}{kr_j}} e^{ikr_j\varphi(\theta_j) - i\pi/4}\left(1 + O\left(\frac{1}{kr_j}\right)\right),$$

where $\alpha(\theta) = -ka^2\cos\theta/\varphi(\theta)$, $\varphi(\theta) = \sqrt{a^2\cos^2\theta + \sin^2\theta}$.

5. Let us now examine the wave field produced by diffraction. In region I, the diffraction field can be represented by (7) in terms of the set of waveguide eigenfunctions.

In region II, the field consists of two qualitatively different types of terms. Terms belonging to the first type, i.e., Ψ_k and Ψ_s, are waves that decrease exponentially from their maximum value on Γ and the free surface with distance from them, and progressive waves in x traveling away from the edge of Γ. The waves Ψ_k exist only for $\mu_0 < 0$, i.e., $\beta c^2/g > 1$, and the waves Ψ_s exist only at a particular frequency: $\omega^2 > \beta g$.

The second type of term in (8) is represented by $\Psi_1{}^d$ and $\Psi_2{}^d$. Because of (4), $\Psi_2{}^d$ can be looked upon as a reflection of the field represented by $\Psi_1{}^d$ from the free surface in accordance with a particular law. The term $\Psi_1{}^d$ describes a wave propagating away from the edge of Γ with amplitude that decreases mostly as $(kr_1)^{-1/2}$ with distance from the edge, where surfaces of constant phase tend to a family of ellipses elongated along Oz and centered on O_1 as kr_1 increases.

The terms $\Psi_1{}^d$ and $\Psi_2{}^d$ are qualitatively similar to those in [2].

We now note that the problem of diffraction of the plane wave $\Psi^0 = -\frac{M_0(k_0)}{2\gamma(k_0)} \cdot e^{ik_0 x - \gamma(k_0)z}$ by the edge of Γ can be solved in precisely the same way. Here, $|k_0| < ka$ and $\gamma(k_0) = -(i/a)\sqrt{k^2a^2 - k_0^2}$. Actually, the formulation of the problem remains the same if the resultant field in the region $\{0<z<h\}$ is denoted by Ψ, whereas in

the region $\{z<0\}$ it is represented by $\Psi_z=\Psi+\Psi^0+\Psi_0{}^R$, where $\Psi_0^R=\frac{\mu_0-\gamma(k_0)}{2\gamma(k_0)}e^{ik_0x+\gamma(k_0)z}$ is the wave reflected from the half-plane Γ. The resultant wave field in region I is then given by (7), but now without the term Ψ_0. Apart from the terms on the right hand side of (8), the resultant field in region II is also described by the terms Ψ^0, $\Psi_0{}^R$, and $\Psi^R=-\frac{M_0(k_0)}{2\gamma(k_0)}R(k_0)e^{ik_0x+\gamma(k_0)(z-2h)}$ (wave reflected from the free surface), combined strictly in accordance with the laws of geometric optics. From the mathematical point of view, the latter result is due to the fact that the quantity k_0 is no longer related to the solution of (3), and this means that the integrands in (4) have a residue at this point.

The last of these problems is a typical ranging problem when a rectilinear barrier is immersed in the liquid, parallel to its free surface.

REFERENCES

1. S. A. Gabov and A. G. Sveshnikov, Problems in the Dynamics of Stratified Liquids [in Russian], Moscow, 1986.
2. A. K. Shatov, Zh. Vychisl. Mat. Mat. Fiz., vol. 24, no. 10, pp. 1548-1556, 1984.
3. B. Noble, Methods Based on the Wiener-Hopf Technique [Russian translation], Moscow, 1962.
4. S. A. Gabov and A. G. Sveshnikov, Dokl. Akad. Nauk SSSR, vol. 265, no. 1, pp. 16-20, 1982.
5. V. V. Varlamov, S. A. Gabov, and A. G. Sveshnikov, Diff. Uravn., vol. 20, no. 12, pp. 2088-2095, 1984.

17 June 1986 Chair of Mathematics