TRANSFORMATION FROM THE MITTAG-LEFFLER EXPANSION TO THE EXPANSION OF THE POTENTIAL OF AN ARBITRARY BODY IN THE REGION OF ITS ANALYTICITY

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A transformation is found between the function $1/(1 - z)$ of a complex variable, defined a two-dimensional stellar region, to the reciprocal distance $1/\Delta$ that is a function of a real variable defined in three-dimensional space. Formulas relating the coefficients of the expansion of $1/\Delta$ in terms of spherical functions with the coefficients of the Mittag-Leffler expansion of the corresponding complex function $1/(1 - z)$ are derived.

The gravitational potential of a planet $V=f\int dm/\Delta$ expressed as a series in terms of spherical functions (Laplace series), which is widely used to analyze satellite data, is definitely valid only outside a sphere S passing through the highest point of the planet, since it is derived on the basis of an expansion of the reciprocal distance $1/\Delta$ into a power series that converges only outside S. Additional analysis is required when we wish to examine the validity of the satellite model of the potential in the interior of S, for example, on the physical surfaces of planets [1-3]. However, even when the Laplace series converges on the physical surface, the representation may not be the optimum representation (in the sense of the rate of convergence) because the system of spherical functions is not orthogonal on the nonspherical surface. Of course, it is possible to transform to an expansion in terms of a set of functions that are orthogonal in a given region of observation (for example, the Lame functions outside an ellipsoid or the spatial analog of the Faber polynomials outside an arbitrary smoothed surface). However, this is a very complicated procedure and the transformation from the Laplace series to it is a very difficult and as yet generally unsolved problem. Moreover, the expansion over spherical functions is very convenient in practical analysis and interpretation of both satellite and terrestrial gravimetric observations.

It is therefore desirable to obtain an expansion for the potential in terms of spherical functions that would converge everywhere outside an arbitrary surface, with coefficients that could be chosen so as to ensure the maximum range of convergence for given regions of observations, and to which it would be easy to transform from the Laplace series. It is clear that this expansion may be obtained not from a power series for the reciprocal distance but from another expansion that converges in an arbitrary region. In a previous paper[4] we proposed to use for this purpose the Mittag-Leffler expansion [5, p. 499] for an analytic function in a stellar region, which differs from a power series by the presence of certain factors $C_k^{(n)}<1$ in front of each coefficient of degree k, which depend on the order n of the approximation and on the parameters of the approximation region.

We shall show below that the corresponding expansion for the reciprocal distance that converges in an arbitrary region differs from the generally adopted expansion in terms of spherical functions (which converges only outside S) by the same factor $C_k^{(n)}$:

$$\frac{1}{\Delta}=\lim_{n\to\infty}\frac{1}{r}\sum_{k=0}^{n}C_k^{(n)}\left(\frac{r'}{r}\right)^k P_k(\cos\psi), \tag{1}$$

where r', r are the radial distances of the point of integration and observation, respectively, and ψ is the angular separation between them. We must now solve two problems, namely, we must find the transformation from the function $1/\Delta$ of a real variable defined in three-dimensional space to a complex function $f(z)$ defined in a two-dimensional stellar region, and we must derive formulas relating the coefficients of the expansion for $1/\Delta$ in terms of spherical functions to the coefficients of the Mittag-Leffler expansion for the corresponding complex function $f(z)$.

The first problem can be solved by transforming in the formula for the reciprocal distance

$$\frac{1}{\Delta}=\frac{1}{r\sqrt{1-2(r'/r)\cos\psi+(r'/r)^2}}$$

to the complex variable $z=(r'/r)e^{i\psi}$. This gives $r/\Delta=|1-z|^{-1}$.

We now consider the manifold described by the point z when the point of integration runs through all the points in the body of a planet T while the point of observation lies in the space external to T. This is a bounded (when $r \neq 0$) and closed (because T is closed) single-connected manifold, i.e., a closed region D that does not contain a single real positive number $z = x \geq 1$ (since $\psi = 0$ for $r > r'$) and is bounded by a closed single-valued curve d whose maximum and minimum radii correspond to the maximum and minimum radii R of the surface of the body T:

$$|z_d|_{\max}=r'_{\max}/r_{\min}=R_{\max}/R_{\min},$$

$$|z_d|_{\min}=R_{\min}/r_{\min}=1.$$

In the above stellar region D, centered on $z = 0$, the function $1/(1 - z)$ is analytic and, consequently, can be represented by a Mittag-Leffler expansion that converges in D [5]:

$$\frac{1}{1-z}=\lim_{n\to\infty}\sum_{k=0}^{n}C_k^{(n)}z^k, \tag{2}$$

where

$$C_k^{(n)}=\alpha^k k!\sum_{q=k}^{n}\frac{|S_q^{(k)}|}{q!}\beta^q, \tag{3}$$

$S_q^{(k)}$ are the Stirling numbers of the first kind, and $0<\alpha<1$, $\beta=1-e^{-1/\alpha}$ are parameters that characterize the region that approximates D.

We shall now outline the solution of the second problem. The convergence of the sequence of analytic functions (2) has as its consequence the convergence of the sequence of moduli:

$$\frac{1}{|1-z|} = \lim_{n\to\infty} \left| \sum_{k=0}^{n} C_k^{(n)} \left(\frac{r'}{r} e^{i\psi}\right)^k \right| = \lim_{n\to\infty} \sqrt{1+S_n}, \tag{4}$$

where $S_n = \sum_{m=1}^{2n} A_m^{(n)}(\psi) \left(\frac{r'}{r}\right)^m$,

$$A_m^{(n)}(\psi) = \sum_{l=\frac{m}{2}-\left\langle\frac{m}{2}\right\rangle} (2-\delta_{0l})\, C^{(n)}_{\frac{m}{2}-l}\, C^{(n)}_{\frac{m}{2}+l} \cos 2l\psi,$$

$$N = \min\left(\frac{m}{2}, \frac{2n-m}{2}\right),$$

and $\langle\ \rangle$ represents the integer part.

Using the generating function for Stirling numbers [6]

$$\left(\ln\frac{1}{1-\beta}\right)^k = k! \sum_{q=k}^{\infty} \frac{|S_q^{(k)}|}{q!} \beta^q \tag{5}$$

together with (3), we find that

$$(C_1^{(n)})^k = C_k^{(n)} + \alpha^k \sum_{q=n+1}^{nk} \frac{k!}{q!} E_q^{(k)} \beta^q, \tag{6}$$

where $0 < E_q^{(k)} < |S_q^{(k)}|$ and $C_k^{(n)} \underset{n\to\infty}{\to} (C_1^{(n)})^k$.

Thus, by substituting $(C_1^{(n)})^k$ in (4) in place of $C_k^{(n)}$, we obtain an expansion that approaches r/Δ from above and, then, by replacing $(C_1^{(n)})^k$ with $C_k^{(n)}$, we obtain an expansion that approaches r/Δ from below. Henceforth we shall replace $C_k^{(n)}$ with $(C_1^{(n)})^k$, throughout. We thus obtain

$$\frac{r}{\Delta} = \lim_{n\to\infty} \sqrt{1+S_n^*}, \tag{7}$$

where

$$S_n^* = \sum_{m=1}^{2n} (C_1^{(n)})^m \mathring{A}_m^{(n)} \left(\frac{r'}{r}\right)^m,$$

$$\mathring{A}_m^{(n)} = \sum_{l=\frac{m}{2}-\left\langle\frac{m}{2}\right\rangle}^{N} (2-\delta_{0l}) \cos 2l\psi.$$

If we begin with (7) for $|S_n^*| < 1$, and expand into a power series, we eventually obtain

$$\frac{r}{\Delta} = \lim_{n\to\infty} \sum_{k=0}^{n} (C_1^{(n)})^k \left(\frac{r'}{r}\right)^k P_k(\psi), \tag{8}$$

where

$$P_k(\psi)=\sum_{i=1}^{k} d_i \sum_{\sum l_j k_j = k;\ l_j \leqslant 2n} (i; k_1, \ldots, k_j)\,(\mathring{A}_{l_1}^{(n)})^{k_1} \ldots (\mathring{A}_{l_j}^{(n)})^{k_j},$$

$$d_i=(-1)^{i+1}\frac{1\cdot 1\cdot 3\cdots(2i-3)}{2\cdot 4\cdots 2i}, \qquad (9)$$

$$(i; k_1, \ldots, k_j)=\frac{i!}{k_1!\ldots k_j!}.$$

We shall now show that (9) defines the classical Legendre polynomial. When $|z|<1$, the analytic function $1/(1 - z)$ can be expanded into a power series of the form

$$\frac{1}{1-z}=\lim_{n\to\infty}\sum_{k=0}^{n} z^k.$$

Proceeding as before, we find that, when $r'/r < 1$,

$$\frac{r}{\Delta}=\frac{1}{|1-z|}=\lim_{n\to\infty}\sum_{k=0}^{n}\left(\frac{r'}{r}\right)^k P_k(\psi).$$

On the other hand, we know that, when $r'/r < 1$, there is the unique expansion

$$\frac{r}{\Delta}=\sum_{k=0}^{\infty}\left(\frac{r'}{r}\right)^k P_k(\cos\psi),$$

where $P_k(\cos\psi)$ is the unnormalized Legendre polynomial. Consequently, it does indeed turn out that $P_k(\psi)=P_k(\cos\psi)$.

If we now consider (7) for $|S_n^*|>1$, we can isolate in S_n^* a constant part C such that $S_n'=(S_n^*-C)/(1+C)<1$ and then expand (7) in S_n'. This again leads to (8).

Replacing $(C_1^{(n)})^k$ with $C_k^{(n)}$ in (8), we obtain the expansion (1) that approaches $1/\Delta$ from below, as required.

Substituting (1) or (8) into the formula for the potential, and integrating over the masses of the planet, we obtain the representation

$$V(r,\varphi,\lambda)=\lim_{n\to\infty} f\sum_{k=0}^{n}\sum_{m=0}^{k}\begin{Bmatrix}C_k^{(n)}\\ (C_1^{(n)})^k\end{Bmatrix}\frac{1}{r^{k+1}}(A_{km}\cos m\lambda+B_{km}\sin m\lambda)\,P_{km}(\sin\varphi),$$

which converges uniformly everywhere outside the physical surface of the planet and differs from the Laplace series by the factors $C_k^{(n)}$ or $(C_1^{(n)})^k$. The presence of the former ensures convergence from below and of the second convergence from above. The size of these factors depends on the form of the physical surface of the planet via the parameter α in (3). The choice of this parameter that ensures optimum representation of the potential will be discussed elsewhere. Here we confine our attention to estimates of the coefficients $C_k^{(n)}$ and of their difference from $(C_1^{(n)})^k$ in a particular order n of the approximation.

From (3), (5), and (6), we have

$$0<(\alpha\beta)^k\leqslant C_k^{(n)}<(C_1^{(n)})^k<\lim_{n\to\infty} C_k^{(n)}=1.$$

		$\alpha = 0,36$		$\alpha = 0,1$	
n	k	$C_k^{(n)}$	$(C_1^{(n)})^k$	$C_k^{(n)}$	$(C_k^{(n)})^k$
2	1	0,51		0,1	
	2	0,12	0,26	0,01	0,01
10	1	0,86		0,29	
	10	0,00002	0,2	10^{-10}	$0,4\cdot10^{-5}$
30	1	0,97		0,40	
	30	10^{-14}	0,4	10^{-30}	10^{-12}
100	1	0,9999		0,52	
	100	10^{-47}	0,99	10^{-100}	$0,3\cdot10^{-28}$

In particular

$$C_n^{(n)} = (\alpha\beta)^n; \quad C_1^n = \alpha \sum_{q=1}^{n} \frac{\beta^q}{q}.$$

More detailed estimates for large n can be obtained using asymptotic estimates for $S_q^{(k)}$. We thus find for $k=0\,(\ln n)$ that

$$1-k\beta^{n+1}\alpha e^{1/\alpha}(\alpha \ln n)^{k-1}/n < C_k^{(n)} < 1-k\beta^n(\alpha \ln n)^{k-1}/(n+1), \tag{10}$$

where

$$(C_1^{(n)})^k - C_k^{(n)} < k\beta^{n+1}\alpha^k e^{1/\alpha} \underset{n\to\infty}{\to} 0$$

and this occurs more rapidly as β decreases, i.e., α increases.

When k ~ n, we find that

$$(\alpha\beta)^k < C_k^{(n)} < (k/2)^{n-k}\alpha^k\beta^n, \tag{11}$$

in which case

$$(C_1^{(n)})^k - C_k^{(n)} \underset{n\to\infty}{\to} 0$$

and this occurs more rapidly for smaller α.

We must now estimate the dependence of $C_k^{(n)}$ on α. It is clear from (10) and (11) that $C_k^{(n)}$ increases with increasing α but, even when α → 1 for k ~ n, the coefficient $C_k^{(n)}$ does not tend to unity because β < 1 - 1/e. Transforming (10) and (11), we find that, for small k,

$$1 - k\alpha^k e^{1/\alpha-(n+1)e^{-1/\alpha}} < C_k^{(n)} < 1 - ke^{-n/e}/(n+1),$$

and when k ~ n we have

$$\alpha^k e^{-ke^{-1/\alpha}} < C_k^{(n)} < (k/2)^{n-k} e^{-k/e}.$$

Analysis of these formulas shows that the difference between the upper limit of $C_k^{(n)}$ and unity decreases with increasing n and decreasing k independently of the dependence on α. On the other hand, the lower limit is determined by α and increases with increasing α more rapidly as n increases and k decreases.

As an illustration, let us consider the values of $C_k^{(n)}$ for certain α, n, and k (see table). It is clear that the presence of these factors results in strong smoothing of the Laplace series, especially for the higher harmonics.

We note in conclusion that, by virtue of the Weierstrass theorem on the differentiation of a uniformly converging sequence of analytic functions, the following sequences will converge uniformly outside the physical surface:

$$\frac{\partial V}{\partial l}(r, \varphi, \lambda) = \lim_{n\to\infty} f \sum_{k=0}^{n}\sum_{m=0}^{k} \begin{Bmatrix} C_k^{(n)} \\ (C_1^{(n)})^k \end{Bmatrix} \frac{\partial}{\partial l}\left[\frac{P_{km}(\sin\varphi)}{r^{k+1}}(A_{km}\cos m\lambda + B_{km}\sin m\lambda)\right].$$

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