THEORETICAL DESCRIPTION OF THERMAL SELF-FOCUSING OF SAW-TOOTH ACOUSTIC WAVES

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The geometric acoustic approximation used to analyze thermal self-focusing of saw-tooth waves. The analysis shows that the results differ significantly from those established previously for harmonic waves.

Thermal self-focusing (TSF) of acoustic waves is a nonlinear effect that arises because the velocity of sound is a function of temperature and of the inhomogeneous heating of the medium that accompanies the absorption of sound. TSF has been analyzed in considerable detail in the case of quasiharmonic acoustic waves, using the analogy with optical self-focusing. This analysis is valid when the nonlinear distortion of the wave form is not significant on the TSF scale (e.g., in highly viscous liquids). A different situation is typical in nonlinear acoustics: an initially sinusoidal wave is distorted as it propagates through the medium and eventually assumes a sawtooth wave form. The thermal self-focusing of such waves is examined below.

We shall describe the acoustic beam by the modified Khokhlov-Zabolotskaya-Kuznetsov equation [1,2]:

$$\frac{\partial}{\partial\tau}\left[\frac{\partial p}{\partial x}-\frac{\delta T}{c_0}\frac{\partial p}{\partial\tau}-\frac{\varepsilon}{\rho_0 c_0^3}p\frac{\partial p}{\partial\tau}-\frac{b}{2\rho_0 c_0^3}\frac{\partial^2 p}{\partial\tau^2}\right]=\frac{c_0}{2}\Delta_\perp p \qquad (1)$$

where p is the acoustic pressure, T is the temperature of the medium averaged over the period of the wave, $\delta=c_0^{-1}(\partial c/\partial T)_p$ is the temperature coefficient of the velocity of sound c, c_0, ρ_0 are the unperturbed velocity of sound and the density of the medium, ε, b are the nonlinearity and dissipation parameters, x is the position coordinate along the beam, $\tau=t-x/c_0$, and $\Delta_\perp=r^{-1}\partial/\partial r(r\partial/\partial r)$ is the Laplace operator for the transverse coordinate r. The temperature field will be calculated from the inhomogeneous thermal conduction equation

$$\frac{\partial T}{\partial t}-\frac{\varkappa}{\rho_0 c_p}\Delta_\perp T=\frac{b}{c_0^4\rho_0^3 c_p}\overline{(\partial p/\partial\tau)^2}, \qquad (2)$$

where the right hand side is averaged over the period of the acoustic wave and describes the transformation of acoustic energy into heat as a result of absorption. In this equation, κ, c_p are the thermal conductivity and specific heat of the medium, and it is assumed that the medium is stationary.

Equation (1) must be solved numerically even with the self-interaction is neglected. It can be simplified still further when the wavelength λ is much smaller than the characteristic size L of a temperature inhomogeneity. Let us substitute $p=p(x, r, \theta=\tau-\psi(x, r)/c_0)$, where $\psi(x, r)$ is the shift of the wave front due to the self-interaction. If we use the geometric acoustics approximation $\lambda/L \to 0$, we find from (1) that

$$\frac{\partial p}{\partial x}-\frac{\varepsilon}{\rho_0 c_0^3} p \frac{\partial p}{\partial \theta}-\frac{b}{2\rho_0 c_0^3}\frac{\partial^2 p}{\partial \theta^2}+\frac{\partial \psi}{\partial r}\frac{\partial p}{\partial r}+\frac{\Delta_\perp \psi}{2} p=0, \quad (3)$$

$$\frac{\partial \psi}{\partial x}+\frac{1}{2}\left(\frac{\partial \psi}{\partial r}\right)^2+\delta T=0. \quad (4)$$

Sawtooth waves can be described by substituting Khokhlov type solutions of the Burgers equation. Over one period the acoustic wave, we have

$$p(x, r, \theta)=A(x, r)\left[\operatorname{th}\left(\frac{\varepsilon A(x, r)\theta}{b}\right)-\frac{\omega\theta}{\pi}\right]. \quad (5)$$

where A is the wave amplitude and $2\pi/\omega$ is its period. We assume that the acoustic Reynolds number is large, so that $b \ll 2\pi\varepsilon A/\omega$. Substituting (5) into (2) and (3), and passing to the limit as $b \to 0$, we obtain the following equations for the thermal self-focusing of sawtooth waves:

$$\frac{\partial A}{\partial x}+\frac{\varepsilon\omega A^2}{\pi\rho_0 c_0^3}+\frac{\partial \psi}{\partial r}\frac{\partial A}{\partial r}+\frac{\Delta_\perp \psi}{2} A=0, \quad (6)$$

$$\frac{\partial T}{\partial t}-\frac{\varkappa}{\rho_0 c_p}\Delta_\perp T=\frac{2}{3\pi}\cdot\frac{\varepsilon\omega A^3}{c_0^4\rho_0^3 c_p}. \quad (7)$$

Equations (4) and (6) can be significantly simplified by using the aberration-free approximation [3]. Let us suppose that the wave front remains spherical at all times, and only its curvature β is varying: $\psi(x, r, t)=\varphi(x, t)+r^2\beta(x, t)/2$. This assumption is justified for a parabolic transverse temperature distribution. In the aberration-free approximation, Eq. (6) has the following solution:

$$\frac{A}{p_0}=\frac{\Phi\left(\frac{r^2}{a^2 f^2}\right)}{f}\left[1+x_p^{-1}\Phi\left(\frac{r^2}{a^2 f^2}\right)\int_0^x\frac{dx'}{f(x', t)}\right]^{-1}. \quad (8)$$

where $p_0=A(x=0, r=0)$, $\Phi(\xi)$ is the transverse structure of the beam for $x = 0$, i.e., $A(x=0, r)=p_0\Phi(r^2/a^2)$, a is the initial radius of the beam, $f(x, t)$ is the running dimensionless radius of the beam, $f(x = 0, t) = 1$, and $x_p=\pi\rho_0 c_0^3/(\varepsilon\omega p_0)$ is the scale of nonlinear absorption of the wave. It is clear from (8) that the wave amplitude in the beam is determined by the behavior of the function $f(x, t)$. It follows from (4) that f is a solution of $\partial^2 f/\partial x^2=\delta T_2$, where T_2 is the coefficient of the quadratic term in the expansion of the temperature in terms of the transverse coordinate, i.e., $T=T_0-T_2 r^2/2+\ldots$. Let $t_0=\rho_0 c_p a^2/(12\varkappa)$ represent the characteristic time necessary for the temperature to settle. The quantity T_2 can readily be found from (7) and (8) for $t \gg t_0$ (stationary TSF) and for $t \ll t_0$ (initial TSF):

$$\frac{\partial^2 f}{\partial z^2}=\frac{x_p}{R_T}\cdot\frac{\operatorname{sign}(\delta)}{f^2}\left[1+\int_0^z\frac{dz'}{f}\right]^{-3} \quad \text{for } t \gg t_0, \quad (9)$$

$$\frac{\partial}{\partial(t/t_0)}\left(\frac{1}{f}\frac{\partial^2 f}{\partial z^2}\right)=\frac{x_p}{R_T}\frac{\operatorname{sign}(\delta)}{f^5}\left[1+\int_0^z\frac{dz'}{f}\right]^{-4} \quad \text{for } t \ll t_0. \quad (10)$$

where $z = x/x_p$, R_T is the characteristic self-interaction scale, i.e., $R_T = \varkappa/|\delta| I_0$ and $I_0 = p_0^2/(3\rho_0 c_0)$ is the intensity on the beam axis for $x = 0$. Equations (9) and (10) cannot be solved analytically, and it is clear that, in terms of the variables z, t/t_0, the solution depends on the single dimensionless parameter x_p/R_T. As in the case of the thermal self-focusing of harmonic waves [3], we can distinguish between the cases of "thin" and "thick" lenses. In the former case, for $x_p/R_T \ll 1$, the beam width changes very slightly over the sound-absorption scale x_p, and only the curvature of the wave front undergoes a change. The attenuated wave is then focused (for $\delta < 0$) at a distance x_f, where $x_f = 2R_T$ for $t \gg t_0$, $x_f = 3R_T t_0/t$ for $t \ll t_0$. For the thick lens, for which the reduction in acoustic energy can be neglected, we have $x_p/R_T \gg 1$ in (9), and $x_p/R_T \gg t_0/t$ in (10). When $t \gg t_0$, it follows from (9) that the beam "collapses" at $x_f = (R_T x_p/2)^{1/2}$. When $t \ll t_0$, it follows from (10) that the TSF scale becomes $x_f = (R_T x_p t_0/2t)^{1/2}$.

The above analysis shows that the thermal self-focusing of sawtooth waves occurs in a different way as compared with harmonic waves [1,3]. Both the ray paths and the behavior of the wave amplitudes are found to be different. The reason for this is the nonlinear character of the absorption of sawtooth waves.

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