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INVARIANT REGULARIZATION OF INFRARED DIVERGENCES IN THE BACKGROUND FIELD METHOD FOR TWO-DIMENSIONAL NONLINEAR THEORIES

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Various methods for separating the ultraviolet and infrared divergences in two-dimensional nonlinear sigma models are discussed. An infrared regularization is proposed that takes into account the ultraviolet renormalization effect in the preceding orders and makes it possible to obtain invariant ultraviolet counterterms by the background field method.

The background field method is an effective means for determining quantum corrections in quantum field theory (see, e. g., [1] and references therein). Its role is particularly important in nonlinear quantum-field models having a complex symmetry structure, in particular, in two-dimensional nonlinear sigma models (TNSMs), which can be regarded as being renormalizable in the generalized sense [2]. In [3, 4] and also in [5], a covariant formalism was developed in explicit form for determining counterterms within the framework of this method in TNSMs, which is used in all subsequent calculations in both the boson and supersymmetric versions.

As is known, the essence of the background field method lies in the following important property. If a quantum field $\varphi(x)$ is split into a classical (background) field $\Phi(x)$ and an additional quantum term $\pi(x)$, then an ordinary effective action is equal to the generating functional of one-particle-irreducible vacuum diagrams in the presence of the background field. In other words, to obtain the generating functional for all one-particle-irreducible diagrams it suffices to calculate only the vacuum diagrams (i. e., without the external lines corresponding to quantum fields), which have external background fields at their vertices.

For a TNSM this simple calculation scheme proves insufficient owing to the nonlinear character of the field splitting into the classical and the quantum parts. It turns out that in this theory, in order to remove all ultraviolet (UV) divergences in higher loops, one must not only add counterterms having the structure of action and interpreted as a renormalization of the target-space metric, but also perform a nonmultiplicative renormalization of the quantum field [6, 7].

The additional term to the two-loop counterterm obtained in [7], which corresponds to the renormalization of the quantum field, is proportional to the equation of motion and has a divergence of the type of ε^{-2} (ε is the parameter of dimensional regularization). In the present paper we shall show that allowance for the quantum field renormalization is also necessary for a correct separation of infrared (IR) and UV divergences in higher orders by adding to the action some invariant terms that regularize the IR divergences. As will be seen later, the corresponding contribution has a singularity of the form of ε^{-1} and turns out to be important for complete canceling out of noninvariant counterterms.

The interest in studying the structure of UV divergences in TNSMs is to a great extent due to the needs of the string theory (see [8] and references therein), which deals with compact world surfaces and is free of IR divergences. However, since the UV counterterms (related to the interaction locality) are independent of global properties of the world surfaces [9], it is more convenient to perform real calculations of the counterterms assuming that the two-dimensional space is flat and infinite [10, 11]. But then one has to deal with the problem of IR divergences characteristic of two-dimensional theories on a flat infinite space.

Indeed, from the expression for the Fourier transform of an ordinary massless propagator $1/k^2$ in the d-dimensional (Euclidean) space

$$F_d\left[\frac{1}{k^2}\right] \equiv (2\pi)^{-d}\int d^dk\frac{\exp\{ikx\}}{k^2-i0} = (4\pi)^{-d/2}\Gamma\left(\frac{d}{2}-1\right)(x^2)^{1-d/2} \qquad (1)$$

it follows that the propagator diverges in the coordinate representation for $d=2$.

A more rigorous analysis of the problem [12-14] reveals the reason for this difficulty. The matter is that the massless propagator satisfying the equation

$$k^2 D(k) = 1 \tag{2}$$

is not determined uniquely but accurate to an arbitrary constant C:

$$D(k) = \frac{1}{k^2} + C\delta^{(d)}(k) \tag{3}$$

or, in the coordinate representation,

$$D(x) = (4\pi)^{-d/2}\Gamma\left(\frac{d}{2}-1\right)(x^2)^{1-d/2} + (2\pi)^{-d}C. \tag{4}$$

For $d > 2$ this constant is fixed ($C = 0$) by the requirement that the propagator should decrease at large distances, while for $d = 2$ there exists arbitrariness that can be used to determine a propagator free of IR divergences. As a propagator of this kind one can take, for instance, the expression $D(x) - D(0)$.

This subtraction of IR divergences is the simplest case of the application of the so-called R^*-operation [15, 16], which supplements the standard R-operation [17] with the corresponding subtraction of IR divergences. In TNSMs, in higher orders of perturbation theory there is an overlap of UV and IR divergences, therefore in the case under consideration the R^*-operation is carried out in a more complicated manner than a simple subtraction of a singularity from a free propagator [18].

However, in the majority of works devoted to the calculation of β-functions in different versions of TNSM, particularly in low orders of perturbation theory, use is made of another method for eliminating IR divergences. Here we make a brief remark. Of course, the introduction of dimensional regularization ($d = 2 - 2\varepsilon$) regularizes not only the UV divergences but also the IR divergences. However, both of them manifest themselves in a similar way, namely as poles with respect to ε. Therefore to separate these divergences a massless term is added to the action (see below).

To make the subsequent formulas more illustrative, we confine ourselves to a simple version of TNSM, namely we consider a boson model with the action

$$S = \frac{1}{2}\int dx\, G_{ij}(\varphi)\partial_\mu\varphi^i\partial^\mu\varphi^j, \tag{5}$$

whose field manifold is a locally symmetric space ($R_{ijkl;n} = 0$). All the conclusions will apply to more complex cases as well. The massive term of the field $\varphi^i(x)$ has the form

$$S_m = \frac{m^2}{2}\int dx\, G_{ij}(\varphi)\varphi^i\varphi^j. \tag{6}$$

The indicated term added to the action is of course of auxiliary character, and, after the UV counterterms are calculated, the number m should be equated to zero.

In the calculation by the background field method the action is expanded in powers of the quantum field whose role in TNSMs is played by the vector $\xi^i(x)$ tangential to the geodesic at the point of the field manifold with the coordinate $\Phi^i(x)$ [4].

The one-loop divergent counterterm obtained from (5) and (6) is equal to

$$\Delta_1 S + \Delta_1 S_m = \frac{1}{4\pi\varepsilon}\int dx\left\{\frac{1}{2}R_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j + \frac{m^2}{6}R_{ij}\varphi^i\varphi^j\right\}. \tag{7}$$

The second term in (7) vanishes for $m \to 0$ and is not significant in the one-loop approximation. However, it makes a contribution to the UV counterterms in the subsequent orders owing to the appearance of integrals which behave like m^{-2}. As a result, after the IR regularization is removed ($m \to 0$), the part of the two-loop counterterm proportional to $1/\varepsilon$ contains, besides the well-known expression

$$\frac{1}{(4\pi)^2\varepsilon}\frac{1}{2}\int dx\, R_{abci}R_j^{abc}\partial_\mu\varphi^i\partial^\mu\varphi^j, \tag{8}$$

the noninvariant term

$$\frac{1}{(4\pi)^2\varepsilon}\frac{1}{18}\int dx\, R_k^{cab} R_{lc}\varphi^k\varphi^l\left(R_{iabj}+\frac{1}{3}\omega_{dia}\omega_{djb}\right)\partial_\mu\varphi^i\partial^\mu\varphi^j, \tag{9}$$

where ω_{dia} are the so-called spin connectivity coefficients that are not covariant objects (their origin is dealt with in, e.g., [4]). The appearance of noninvariant counterterms indicates that the applied method for separating the UV and IR divergences arising in the given order is not quite correct.

Expression (9) is obtained by using the IR regularization (6), which itself is noninvariant since $\varphi^i(x)$, in contrast to $\partial_\mu\varphi^i(x)$, is not transformed as a vector on the field manifold. Therefore the noninvariant terms in the counterterms are usually neglected under the assumption that if an invariant IR regularization is introduced, then no such terms must appear [4]. However, in this case the question of existence of such a regularization remains open.

As will be shown by a direct calculation, to separate the IR and UV divergences in higher orders in an invariant manner it is not sufficient to have just an invariant massive term. It is also necessary that the regularization of the IR divergences of a given order be carried out after the UV divergences of the preceding orders in the massive term are removed. Indeed, one can see, that the choice of the IR regularization in the simplest invariant form

$$S_m=\frac{m^2}{2}\int dx\, G_{ij}(\Phi)\xi^i\xi^j \tag{10}$$

(recall that $\xi^i(x)$ is a vector on the field manifold) results in the appearance of additional noninvariant structures in the two-loop counterterm:

$$\frac{2}{(4\pi)^2\varepsilon}\int dx\,\frac{1}{36}R^{ab}\omega_{cia}\omega^c_{jb}\partial_\mu\varphi^i\partial^\mu\varphi^j. \tag{11}$$

An analysis of the origin of these terms shows that they arise from diagrams whose momentum integrals reduce to products of the form of $m^2I_1I_2$ and $m^4I_1I_3$, where

$$I_\alpha\equiv\int\frac{d^dk}{(2\pi)^d}(k^2+m^2)^{-\alpha}=(4\pi)^{-d/2}(m^2)^{d/2-\alpha}\frac{\Gamma(\alpha-d/2)}{\Gamma(\alpha)}. \tag{12}$$

Since I_1 has an UV pole $1/\varepsilon$ and I_2 and I_3 are singular in the IR limit, the superposition of UV and IR divergences takes place here. The indicated terms are not canceled out here owing to the fact that no complete elimination of UV divergences of the preceding orders has been carried out in the IR-singular parts of the diagrams.

Resorting to the example of the application of the R^*-operation in TNSMs [18] one can draw the conclusion that the term regularizing the IR divergences must take into account the renormalization of all the UV subdivergences. An expression of this kind is

$$S_m=\frac{m^2}{2}\int dx\, G^R_{ij}(\Phi)\xi^i_R\xi^j_R, \tag{13}$$

where expressions renormalized with an accuracy of up to the preceding order are substituted for G^R_{ij} and ξ^i_R in every specified order of perturbation theory. In particular, for the IR regularization of the two-loop counterterm one should set

$$G^R_{ij}=G_{ij}-\frac{1}{4\pi\varepsilon}R_{ij}, \tag{14}$$

$$\xi^i_R=\xi^i+\frac{1}{4\pi\varepsilon}\frac{2}{3}R^i_j\xi^j. \tag{15}$$

Direct calculations show that in this case there occurs complete canceling out of noninvariant terms in the two-loop counterterm. In view of the analogy with the R^*-operation, it should be expected that this property will be retained in all orders of perturbation theory.

Thus, expression (13) is the desired IR regularization providing the correct invariant separation of the IR and UV divergences in TNSMs.

REFERENCES

1. L. Abbott, *Nucl. Phys.*, vol. B185, p. 189, 1981.
2. D. Friedan, *Ann. Phys.*, vol. 163, p. 318, 1985.
3. J. Honerkamp, *Nucl. Phys.*, vol. B36, p. 130, 1972.
4. L. Alvarez-Gaume, D. Z. Freedman, and S. Mukhi, *Ann. Phys.*, vol. 134, p. 85, 1981.
5. S. Mukhi, *Nucl. Phys.*, vol. B264, p. 640, 1986.
6. B. L. Voronov and I. V. Tyutin, *Yadernaya Fizika*, vol. 33, p. 1137, 1981.
7. P. S. Howe, G. Papadopoulos, and K. S. Stelle, *Nucl. Phys.*, vol. B296, p. 26, 1988.
8. M. Green, J. Schwarz, and E. Witten, *Superstring Theory*, Cambridge Univ. Press, Cambridge, 1987.
9. G. Keller and R. Silvotti, *Ann. Phys.*, vol. 183, p. 269, 1988.
10. C. G. Callan, D. H. Friedan, E. J. Martinez, and M. J. Perry, *Nucl. Phys.*, vol. B262, p. 593, 1985.
11. G. Curci and G. Paffuti, *Nucl. Phys.*, vol. B286, p. 399, 1987.
12. F. David, *Comm. Math. Phys.*, vol. 81, p. 149, 1981.
13. M. Grignani and M. Mintchev, *Phys. Rev.*, vol. D38, p. 3163, 1988.
14. J. L. Miramontes and J. M. Sanchez de Santos, *Phys. Lett.*, vol. 246B, p. 399, 1990.
15. K. G. Chetyrkin and F. V. Tkachov, *Phys. Lett.*, vol. 114B, p. 133, 1982.
16. V. A. Smirnov and K. G. Chetyrkin, *Teor. i Matem. Fizika*, vol. 63, p. 462, 1985.
17. N. N. Bogolyubov and D. V. Shirkov, *Introduction to Quantum Field Theory* (in Russian), Moscow, 1984.
18. M. T. Grisaru, D. I. Kazakov, and D. Zanon, *Nucl. Phys.*, vol. B287, p. 189, 1987.

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