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FINITENESS OF THE EFFECTIVE ACTION IN THE CHERN-SIMONS THEORY AND THE CONFORMAL INVARIANCE

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The effective action in the gauge theory for the (2+1)-dimensional space-time was considered and its dilatational invariance was proved. Based on the dilatational invariance, we demonstrated the finiteness of the effective action in all orders of perturbation theory.

1. Interest in the theory of gauge fields for the (2+1)-dimensional space-time whose Lagrangian includes the Chern-Simons topological invariant is due, on the one hand, to investigations in the theory of gauge fields at high (infinite) temperature [1] and, on the other hand, to the close connection between these models and the topological field theory [2] that implies a correspondence between the (1+1)-conformal theories and the Chern-Simons theory [3].

In a number of recent papers [4-6] one- and two-loop contributions to the effective action have been considered for the theory with the original Lagrangian

$$\mathcal{L} = \frac{\kappa}{4\pi}\varepsilon^{\mu\nu\alpha}\Big\{ A^a_\mu \partial_\nu A^a_\alpha + \frac{1}{3} f^{abc} A^a_\mu A^b_\nu A^c_\alpha \Big\}. \tag{1}$$

These papers considered various regularization schemes for divergent integrals: the dimensional regularization*) [4], the regularization by higher covariant derivatives and the gauge-invariant version of the Pauli-Willards procedure [5], and the introduction of Feynman's truncating factor $\Lambda^2/(k^2+\Lambda^2)$ [6] (for finite Λ it violates the gauge invariance of the theory).

The development of the perturbative scheme in [5] resulted in the finite renormalization $\kappa \to \kappa + C_\nu$, and in [6] it was shown that quantum corrections to two- and three-point Green's functions did not exist at all, at least in the two-loop approximation. The one-loop effective action turned out to be finite after the regularization removal [5, 6].

In this short paper we discuss the conformal invariance of the theory with original Lagrangian (1) and the problem of finiteness of the effective action.

2. The conformal invariance of the classical theory with Lagrangian (1) results in a zero trace of the canonical energy-momentum tensor

$$T_{\mu\nu} = -\varepsilon^{\beta\alpha}_{\mu} A^a_\beta \, \partial_\nu A^a_\alpha - \eta_{\mu\nu}\mathcal{L}, \quad T^\mu_\mu = -\frac{1}{2}\varepsilon^{\mu\nu\alpha} A^a_\mu F^a_{\nu\alpha}, \tag{2}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{3}$$

It is clear that T^μ_μ vanishes on the equations of motion

$$F^a_{\mu\nu} = 0. \tag{4}$$

After the calculation of one-loop contributions there can appear conformal anomalies in the theory if use is made of a regularization not preserving the conformal invariance (and even when the regularization preserves the conformal invariance [7]). The Pauli-Willards regularization and Feynman's truncation are

*) In our opinion, in 3-dimensional theories the dimensional regularization leads to results that do not coincide with those obtained by conventional methods, e.g., by Feynman's truncation.

sure to violate the conformal invariance. However, we shall show that there are no conformal anomalies in the theory with action (1).

3. The action in the Chern-Simons theory, which includes a term fixing the gauging and ghosts, is taken in the form

$$S=\int d^3x\Big\{\frac{1}{2}\varepsilon^{\mu\nu\alpha}\Big(A^a_\mu\partial_\nu A^a_\alpha+\frac{g}{3}f^{abc}A^a_\mu A^b_\nu A^c_\alpha\Big) \\ +\partial_\mu A^{a\mu}B_a+\bar{c}^a\partial_\mu(D^\mu c)^a\Big\}. \tag{5}$$

This action is conformally invariant if the following massive field dimensions are chosen:

$$d_A=1,\quad d_B=1,\quad d_{\bar c}=d_c=\frac{1}{2}. \tag{6}$$

Formula (5) yields the expressions

$$D_{ab}(p)=\frac{i}{p^2+i\varepsilon}\delta_{ab} \tag{7}$$

and

$$D^{\mu\nu}_{ab}=\frac{\varepsilon^{\mu\alpha\nu}p_\alpha}{p^2+i\varepsilon}\delta_{ab} \tag{8}$$

for the ghost and gluon field propagators, respectively.

We shall calculate the degree of divergence of the diagrams in this theory. As usual, if a diagram contains I_A internal gauge field lines, I_c internal ghost field lines, and V vertices containing δ_v momenta each, then

$$\omega(G)=3L-I_A-2I_c+\sum_{v=1}^{V}\delta_v, \tag{9}$$

where L is the number of loops. Using the fact that

$$L=I_A+I_c+1-V \tag{10}$$

and

$$I_A=\frac{1}{2}\Big(\sum_v A_v-E_A\Big),\quad I_c=\frac{1}{2}\Big(\sum_v C_v-E_c\Big), \tag{11}$$

where $A_v(C_v)$ is the number of gauge (ghost) field lines arriving at the vertex v and $E_A(E_c)$ is the number of external gauge (ghost) lines, we obtain

$$\omega(G)=3+\sum_v\Big(\delta_v+A_v+\frac{C_v}{2}-3\Big)-E_A-\frac{E_c}{2}. \tag{12}$$

In the theory with action (5) there are two vertices:


, for which $\delta_v=0$, $A_v=3$, $C_v=0$,

and


, for which $\delta_v=0$, $A_v=1$, $C_v=2$.

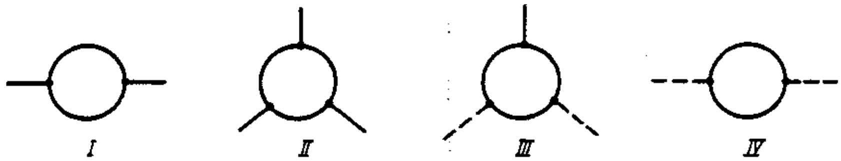
For both of them the vertex index is $\omega_v\equiv\delta_v+A_v+C_v/2-3=0$, and consequently we have

$$\omega(G)=3-E_A-E_c/2. \tag{13}$$

In view of the possibility of transferring the differentiation to the external c-line after discarding the total divergence in the Lagrangian, one can reduce $\omega(G)$ by $E_c/2$, whence

$$\omega(G) = 3 - E_A - E_c. \tag{14}$$

Only the following four types of diagrams are divergent:



We use Feynman's truncation for the regularization and redefine the propagators as

$$\frac{1}{p^2} \to \frac{1}{p^2} - \frac{1}{p^2+\Lambda^2} = \frac{1}{p^2} \cdot \frac{\Lambda^2}{p^2+\Lambda^2}.$$

This is equivalent to the following modification of the part of action (5) quadratic with respect to A and c:

$$S^q \Rightarrow S^\Lambda = \int d^3x \Big\{ \frac{1}{2}\varepsilon^{\mu\nu\alpha} A^a_\mu \partial_\nu \Big(1 + \frac{\Box}{\Lambda^2}\Big) A^a_\alpha + \bar{c}^a \Box \Big(1 - \frac{\Box}{\Lambda^2}\Big) c^a \Big\}. \tag{15}$$

It is clear that expression (15) is not conformally invariant for a finite Λ. We shall prove that after the regularization is removed the effective action becomes conformally invariant.

4. We represent the one-loop effective action in the form of a sum:

$$\Gamma^{(1)\Lambda}_{\text{eff}} = \Gamma^{(1)\Lambda}_3 + \Gamma^{(1)\Lambda}_4, \tag{16}$$

where $\Gamma^{(1)\Lambda}_3$ is part of $\Gamma^{(1)\Lambda}_{\text{eff}}$ which is no more than cubic with respect to A, $\bar{c}$, and c, and $\Gamma^{(1)\Lambda}_4$ denotes all the other contributions. It is shown in [6] that $\lim\limits_{\Lambda\to\infty} \Gamma^{(1)\Lambda}_3 = S$, where S is action (5) (in contrast to [6], we have taken into account the "classical" fields $\bar{c}$ and c). The term $\Gamma^{(1)\Lambda}_4$ is finite for $\Delta \to \infty$, and we have $\lim\limits_{\Lambda\to\infty} \Gamma^{(1)\Lambda}_4 = \Gamma^{(1)}_4$ because all diagrams involved in $\Gamma^{(1)}_4$ are finite and do not require regularization. We now note that with conformal mappings the expressions $\Gamma^{(1)\Lambda}_3$ and $\Gamma^{(1)\Lambda}_4$ are transformed independently. The Chern-Simons action is dilatationally invariant; the action S is also conformally invariant because only the conformally invariant theory (5) was used for its calculation. Consequently, for $\Lambda \to \infty$ the expression $\Gamma^{(1)\Lambda}_{\text{eff}}$ tends to a finite conformally invariant limit $S \equiv \Gamma^{(1)}_{\text{eff}}$.

Thus, in the one-loop approximation Γ_{eff} is dilatationally (conformally) invariant, and consequently there are no conformal anomalies, that is

$$\langle T^\mu_\mu \rangle = \partial_\mu \mathcal{D}^\mu = 0, \tag{17}$$

where $\mathcal{D}_\mu$ is the dilatational current.

By virtue of the Adler-Bell-Jackiw (ABJ) theorem [8], if there are no anomalies in the one-loop approximation, then the theory has no anomalies at all. Hence, relation (17) is retained in any order with respect to $\hbar$, and the total effective action is conformally invariant. (We have retained the ghost fields in the "classical" action so that it be possible to apply the ABJ theorem [8].)

Note that the above proof of the conformal invariance of the theory is also applicable when the regularization proposed in [5] is used, because the only requirement for the proof is that, after the regularization is removed, the expression $\Gamma^{(1)}_3$ be finite and have the form

$$\Gamma^{(1)}_3 = \int d^3x \Big\{ \frac{\alpha}{2}\varepsilon^{\mu\nu\alpha} \Big[A^a_\mu \partial_\nu A^a_\alpha + \frac{1}{3} f^{abc} A^a_\mu A^b_\nu A^c_\alpha \Big] + \frac{1}{2}\partial_\mu A^{a\mu} B_a + \bar{c}^a \partial_\mu (D^{\mu c})^a \Big\},$$

where α is a constant.

5. A direct consequence of the conformal invariance is the finiteness of the effective action in the theory with original Lagrangian (5) in all loops. Indeed, the effective action $\Gamma_{\text{eff}}(A, \bar{c}, c, B, \mu)$ depends on the gauge fields, the field B, and the ghost fields, and also on μ, which is the subtraction point in the renormalized theory. However, in the case under consideration the effective action is conformally invariant, and the theory involves no classical dimensional parameters. Therefore, the effective action cannot depend on μ and, hence, it cannot be only finite. This example demonstrates a close relationship between the conformal invariance and the finiteness of the effective action. It seems that for the action to be finite it is necessary that the following two conditions be fulfilled: the absence of conformal anomalies and the finiteness of the effective action in at least a single loop. As is well known, in Einstein's 4-dimensional gravitation one of these requirements is violated and in the Yang-Mills theory both of them do not hold.

In conclusion we should like to stress that the topological field theories (see [2]), which include the Chern-Simons theory considered, may prove to be finite theories. This problem will be discussed in detail in our further publications.

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