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DETERMINATION OF QCD CONDENSATES FROM THE SUM RULES BASED ON ANALYTICAL PROPERTIES OF THE PION FORMFACTOR

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Based on the analytical properties of the pion formfactor and the sum rules of quantum chromodynamics (QCD), the values of the gluon and the four-quark condensates are determined. The nonperturbative structure of the pion formfactor leads to an increase in the condensate values as compared to the standard ones.

1. INTRODUCTION

Since the publication of the paper by Shifman, Vainstein, and Zakharov [1] much attention has been given to the problem of determining the values of the QCD condensates [2-7]. However, the spread of the results is rather wide. In the determination of the ρ-meson parameters by means of the sum rules in [1] use was actually made of the zero width approximation. Recently I have derived a simple formula for the pion formfactor, which provides a good description of the ρ-meson mass and width and gives the values of the mean-square pion radius and scattering length agreeing with experiment [8]. Moreover, this representation of the pion formfactor admits of a simple separation of the nonperturbative contribution, which is very convenient for the application in the QCD sum rules.

In this connection, it is interesting to solve the problem inverse to the one solved in [1]: to determine the values of the QCD condensates involved in the right-hand sides of the sum rules with the aid of the known representation of the pion formfactor, and this is the subject of the present paper.

2. THE REPRESENTATION OF THE PION FORMFACTOR AND THE UNIFORMIZING VARIABLE

The dominant contribution to the pion formfactor $F_\pi(t)$ for $|t| \leqslant 10$ GeV^2 comes from the ρ-meson [7]. We shall assume that in $F_\pi(t)$ the ρ-meson is characterized by the following four quantities: the mass m_ρ, the width Γ_ρ, the scattering length a_1^1, and the mean-square pion radius $\langle r_\pi^2 \rangle$. The formfactor $F_\pi(t)$ is the boundary value of a function $F(t)$ analytic in the complex plane t with the cut $[4m_\pi^2, \infty)$:

$$F_\pi(t) = \lim_{\varepsilon \to \infty} F(t + i\varepsilon), \quad t \in [4m_\pi^2, \infty).$$

In the interval $4m_\pi^2 \leqslant t < 16m_\pi^2$ the unitarity condition receives a contribution only from the two-particle state, but we shall assume that the two-particle unitarity condition is fulfilled throughout the cut. This approximation leads to a simple proof of the existence of the cut $(-\infty, 0]$ for the function $F(t)$ on the second sheet of its Riemann surface. This cut appears owing to the cross channel of the isovector p-wave scattering amplitude [7]. Finally, $F(t)$ is a meromorphic function on the four-sheeted Riemann surface:

$$F(t) = C/[(\omega - \omega_1)(\omega + \omega_1^*)(\omega - \omega_2)(\omega + \omega_2^*)], \tag{1}$$

where the uniformizing variable has the form

$$\omega(t) = (iq - 1)^{1/2}, \quad q = (t/4m_\pi^2 - 1)^{1/2}. \tag{2}$$

In the complex plane ω the image of the real axis $\operatorname{Im} q = 0$ consists of two hyperbolas:

$$v^2 - u^2 = 1, \quad uv = q/2, \tag{3}$$

where $u = \mathrm{Re}\,\omega$, $v = \mathrm{Im}\,\omega$.

In the approximation of zero ρ-meson width the poles ω_1 and $-\omega_1^*$ are located on the hyperbola $v > 0$ symmetrically relative to the axis $u = 0$. The poles ω_2 and $-\omega_2^*$ are also symmetric relative to this axis and lie on the hyperbola $v < 0$. The requirement that, in view of the normalization condition $F_\pi(t)|_{t=0} = 1$ following from the definition of the pion electric charge, the values of Γ_ρ, a_1^1, and $\langle r_\pi^2\rangle$ calculated with the aid of (1) and (2) should coincide with experimental values leads to fixation of the problem parameters [8]. Here we use the values of the parameters ω_1 and ω_2 obtained under the assumption of 4% accuracy of the four experimental characteristics of the ρ-meson:

$$\begin{aligned} \omega_1 = u_1 + iv_1: &\quad u_1 = 0.96 \pm 0.02, \quad v_1 = 1.28 \pm 0.03, \\ \omega_2 = u_2 + iv_2: &\quad u_2 = 1.63 \pm 0.03, \quad v_2 = 1.53 \pm 0.05. \end{aligned} \tag{4}$$

In this case the poles ω_1 and $-\omega_1$ are slightly displaced from the hyperbola owing to the smallness of the ratio Γ_ρ/m_ρ, so that this can be interpreted as a perturbative effect. The poles ω_2 and $-\omega_2$ recede far from the hyperbola, and their shift relative to the hyperbola can reasonably be regarded as a nonperturbative effect.

3. SEPARATING OUT THE NONPERTURBATIVE CONTRIBUTION TO THE RIGHT-HAND SIDE OF THE SUM RULES

To determine the QCD condensates we use the well-known sum rules [1]:

$$\frac{2}{3}M^{-2}\int ds\,\exp\Big\{-\frac{s}{M^2}\Big\}R^{I=1}(s) = \Big[1 + \frac{\alpha_s(M)}{\pi} - \frac{2\pi^2 f_\pi^2 m_\pi^2}{M^4} + \frac{\pi^2}{3}\frac{C_1}{M^4} - \frac{896\pi^3}{81}\frac{C_2}{M^6}\Big], \tag{5a}$$

$$\frac{2}{6}M^{-4}\int ds^2\,\exp\Big\{-\frac{s}{M^2}\Big\}R^{I=1}(s) = \Big[1 + \frac{\alpha_s(M)}{\pi} + \frac{2\pi^2 f_\pi^2 m_\pi^2}{M^4} - \frac{\pi^2}{3}\frac{C_1}{M^4} + \frac{2\cdot 896\pi^3}{81}\frac{C_2}{M^6}\Big], \tag{5b}$$

where

$$R^{I=1} = \frac{\sigma(e^+e^- \to \text{hadrons},\ I=1)}{\sigma(e^+e^- \to \mu^+\mu^-)},$$

$$C_1 = \Big\langle\frac{\alpha}{\pi}GG\Big\rangle \text{ is the gluon condensate,}$$

$$C_2 = \langle\alpha\bar{q}q\rangle^2 \text{ is the four-quark condensate.}$$

As in [1], confining ourselves to the case

$$\sigma(e^+e^- \to \text{hadrons},\ I=1) = \sigma(e^+e^- \to \pi^+\pi^-),$$

we find

$$R^{I=1}(s) = \frac{1}{4}[(s - 4m_\pi^2)/s]^{3/2}|F_\pi(s)|^2. \tag{6}$$

When substituting (6) into (5) we use the fact that the displacement of the poles ω_1 and $-\omega_1$ from the hyperbola $\mathrm{Im}\,\omega > 0$ is small. The integration over the physical region turns in the plane ω into integration along the positive branch $\mathrm{Im}\,\omega > 0$ of the hyperbola. In the integration over the hyperbola, the terms linear with respect to the deviation of the pole ω_1 from the hyperbola vanish by virtue of the relation $v dv - u du = 0$, which makes it possible to use the well-known representation of the δ function [9]:

$$\lim_{a\to 0}\frac{a}{a^2 + x^2} = \pi\delta(x). \tag{7}$$

As a result, we obtain approximate analytical expressions for the integrals on the left-hand sides of the sum rules:

$$\frac{2}{3}M^{-2}\int ds\,\exp\Big\{-\frac{s}{M^2}\Big\}R^{I=1}(s) = \Phi(C, u_1, v_1, u_2, v_2), \tag{8a}$$

$$\frac{2}{6}M^{-4}\int ds^2\,\exp\Big\{-\frac{s}{M^2}\Big\}R^{I=1}(s) = 4m_\pi^2(2u_1^2 + 1)^2\Phi(C, u_1, v_1, u_2, v_2), \tag{8b}$$

where

$$\Phi = \frac{16m_\pi^2C^2}{\delta}\cdot\frac{[4u_1^2(u_1^2+1)]^{3/2}}{(1+2u_1^2)^2}\cdot\exp\Big\{-\frac{4m_\pi^2}{M^2}(1+2u_1^2)\Big\}\Big/\{4u_1^2[(u_1-u_2)^2 + (v_1-v_2)^2][(u_1+u_2)^2+(v_1-v_2)^2]\},$$

and $\delta=(du^2+dv^2)^{1/2}$ is the distance from the pole ω_1 to the nearest point on the hyperbola. The presented procedure is in fact the separation of the nonperturbative contribution from the ρ-meson to the right-hand sides of sum rules (5).

4. THE DETERMINATION OF QCD CONDENSATES

Using the parameter values (4) in expressions (8) one can determine the values of the gluon and the four-quark QCD condensates, which in this approach enter into the right-hand sides of sum rules (5) as free parameters. Then, following [1], we include the contribution from the continuum in the following way:

$$R^{I=1}_{\rm cont}(s)=\frac{3}{2}\Big(1+\frac{\alpha_s(s)}{\pi}\Big)\cdot\theta(s-s_0). \tag{9}$$

In [1] the value of $s_0 = 1.5$ GeV^2 was used, but we shall regard s_0 as an additional free parameter subject to determination from the sum rules. We determine the free parameters C_1, C_2, and s_0 from the condition of the best coincidence of functions (8a) and (5a) for 0.4 GeV$^2 \leqslant M^2 \leqslant 1$ GeV^2 and the best coincidence of functions (8b) and (5b) for 0.5 GeV$^2 \leqslant M^2 \leqslant 0.7$ GeV^2 [1]. The calculations were carried out using the χ^2 minimization program FUMILI elaborated in the United Institute for Nuclear Research. As a result, the following values were obtained for the QCD condensates:

$$\Big\langle\frac{\alpha}{\pi}GG\Big\rangle = (8.3\pm1.4)\times10^{-2}\ \text{GeV}^4,$$
$$\alpha_s\langle\bar qq\rangle^2=(2.7\pm0.4)\times10^{-4}\ \text{GeV}^6.$$

The parameter s_0 was determined less accurately, and it turned out to be

$$s_0 = 3.75\pm2.25\ \text{GeV}^2.$$

The resulting value of the gluon condensate is about 7 times as great as the "standard value" given in [1], and the value of the four-quark condensate practically coincides with the "standard value" from [1]. This can be regarded as an indication of the fact that the estimate for the four-quark condensate in [1] is based on a more fundamental and model-independent assumptions than the estimate for the gluon condensate. As a consequence of the error in the parameter s_0, the majority of the estimates in the literature [1, 2] fall within this interval.

5. CONCLUSION

The application of a model for $R^{I=1}(s)$ more accurate than the one in [1] leads to an increase in the values of the QCD condensates of dimensions 4 and 6. This result is in qualitative agreement with the data from [2], and for the estimate of the gluon condensate the agreement is also quantitative. The value of the gluon condensate also agrees with that obtained within the framework of a model with an infinite number of vector mesons [7].

An improvement of the values of the condensates studied by an analogous method can be achieved by taking into account the radial excitation of the ρ-meson. The next step in this direction will be an allowance for the ρ'-meson, which will probably make it possible to investigate condensates of higher dimensions.

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