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ON THE STATISTICAL THEORY OF STRENGTH OF BRITTLE SOLIDS

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The probability of enhanced concentration of microdefects in a small volume, irrespective of its shape, was determined. This and a Griffith's type representation were used to determine the probability of rupture depending on the stress applied and to analyze the size effect and the dependence of ultimate strength on the average concentration of defects.

This paper considers a model of brittle fracture. Due to a random distribution of defects throughout the bulk of a solid, a certain number of them may accumulate in a fairly small region. The resulting weak area, under a certain external stress, gives rise to growth of a crack, which ultimately leads to the specimen rupture. The statistical character of brittle fracture is supported by the spread of the ultimate strength values observed when testing identical specimens [1].

Thus, the determination of strength reduces to calculating the probability of the presence of a weak area in the specimen which becomes unstable under a given stress.

Let us consider a solid with a volume V in which a homogeneous field of tensile stresses p is produced. Assume that N identical defects are distributed randomly in this body and that the defects are noninteracting within their effective volume. Consequently, a defect can be found at any point of the solid with an equal probability. In this case, m defects nucleated in a volume Δ lead, at a given load p, to a specimen fracture. In other words, the probability of fracture W_m is the probability that m out of N defects, randomly distributed throughout the volume V, will get into the volume $\Delta \ll V$. This probability is found in Supplement (S.11).

If $N \gg m$, we have $C_N^m = N^m/m!$. Substituting this expression into (S.11) with an accuracy up to the main term in the expansion W_m into power series Δ/V, we obtain

$$W_m = \frac{m^3 N^m}{m!}\left(\frac{\Delta}{V}\right)^{m-1} = \frac{m^3}{m!}(n\Delta)^m \frac{V}{\Delta}, \tag{1}$$

where $n = N/V$ is the average number density of the defects.

In [2, 3], where similar problems were considered, an expression for W_m was derived that differs from (1) in the absence of m^3. The technique of partitioning the volume V into volumes Δ used in these works admitted of skipping dangerous situations. The method of determining W_m used in this paper (see Supplement) does not depend on partitioning V into Δ, and this accounts for the larger W_m.

The conditions of fracture are found in terms of critical local concentration of defects

$$n_{cr} = m/\Delta. \tag{2}$$

Using for $m!$ Stirling's formula, we obtain

$$W_m = \frac{n_{cr} V}{\sqrt{2\pi}}\left(e\,\frac{n}{n_{cr}}\right)^m m^{3/2} \tag{3}$$

from (1) and (2). It is convenient to introduce the notation

$$\beta = \ln\frac{n_{cr}}{n} - 1. \tag{4}$$

From (3) and (4) it follows that

$$W_m = \frac{n_{cr} V}{\sqrt{2\pi}} \exp\{-\beta m\} m^{3/2}. \tag{5}$$

Henceforth we assume that n and n_{cr} are independent of p. To find the relation $m = m(p)$, we make the assumption that fracture occurs if m defects are located in virtually one cross section of the specimen. In this case the area S_{cr} taken by the defects differs from l_{cr}^2 by a numerical factor of the order of unity (here l_{cr} is the size of the crack critical for p, i.e., cracks with a diameter $d \geqslant l_{cr}$ are unstable at a given stress). Thus,

$$\Delta = hl_{cr}^2, \tag{6}$$

where h is the effective thickness of the defect layer.

A further propagation is possible if there is a certain relation between the fracture stress and the critical size of the crack. We take this relation to be

$$pl_{cr}^s = \text{const} = k. \tag{7}$$

Our choice is based on the fact that, according to [4], $s = 1/2$ for the brittle fracture. On the other hand, in rupturing of a liquid on a solitary bubble $s = 3/2$ [5]. It can be assumed that for bodies with a low magnitude of shear and, hence, a Poisson's ratio close to 0.5, s is close to 3/2. Therefore, in what follows we assume the range of possible variations to be $3/2 > s \geqslant 1/2$. The relation $m = m(p)$ is found from (2), (6), and (7):

$$m = \left(\frac{p_0}{p}\right)^{2/s}, \tag{8}$$

where $p_0 = (n_{cr}h)^{s/2}k$. The substitution of (8) into (5) yields the dependence of the probability of fracture on the applied stress:

$$W(p) = \frac{n_{cr}V}{\sqrt{2\pi}}\left(\frac{p_0}{p}\right)^{3/s} \exp\left\{-\beta\left(\frac{p_0}{p}\right)^{2/s}\right\}. \tag{9}$$

Its shape is presented in Fig. 1. The function $W(p)$ for the above s rapidly tends to zero as $p \to 0$, has a maximum at $p_1 = p_0(2\beta/3)^{s/2}$ and then decreases as $p \to \infty$. However, $W(p)$ is physically meaningful only for $p < R$, where R is determined from the equation

$$W(R) = 1. \tag{10}$$

One can reasonably assume R to be the tensile strength.

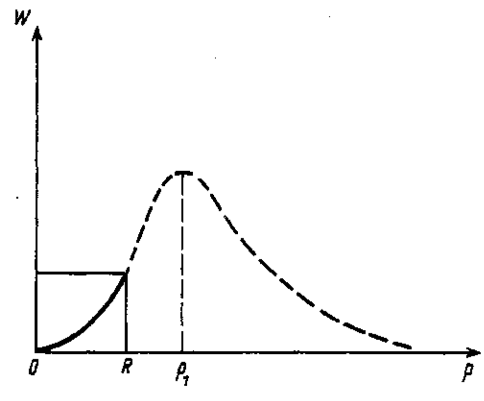

Fig. 1

To solve Eq. (10), we take the logarithm of it, make use of (8) for brevity, and introduce the notation $\mu = \ln(n_{cr}V/\sqrt{2\pi})$, then

$$\mu + \frac{3}{2}\ln m - \beta m = 0. \tag{11}$$

Neglecting the second term in (11), we have

$$m = \mu/\beta. \tag{12}$$

It can easily be seen that solution (11) can be represented as $(\mu/\beta)(1+\delta)$, where $\delta \simeq \ln(\mu/\beta)/[(2/3)\mu-1] \ll 1$. Thus, to within a good accuracy, the rupture strength is

$$R = p_0\left(\frac{\beta}{\mu}\right)^{s/2} = p_0\left[\frac{\ln(n_{cr}/n)-1}{\ln(n_{cr}V/\sqrt{2\pi})}\right]^{s/2}. \tag{13}$$

This solution is of course true for a fairly large number of defects, viz., $nV \gg 1$ and sufficiently large deviations of the local number density of defects from an average one, viz., $n_{cr}/n \gg 1$. In accordance with (13), the ultimate strength diminishes as a logarithm with increasing specimen volume. Note that this result somewhat differs from the earlier obtained dependences of the form C/V [2, 6] or $C_1 + C_2/V$ [1, 3] which are consistent with experiment on specimens with small dimensions ($\sim 10^{-2}$ cm) [1]. However, even for specimens with the cross section of the order of a centimeter, these formulas yield too strong a dependence of the ultimate strength on the volume [6]. Relation (13) is devoid of this disadvantage but for a comparison with experiment one needs data for a wide range of specimen sizes.

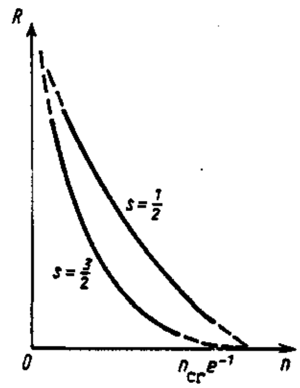

Fig. 2

From our standpoint, of significant interest is the dependence of the ultimate strength on the average concentration of defects (Fig. 2). The ultimate strength decreases with increasing concentration, and this is sufficiently apparent qualitatively. Based on this, an acoustic nonlinear diagnosis of the strength of brittle solids was suggested in [7, 8], which was experimentally confirmed fairly well for structurally imperfect materials, such as concrete. However, an analysis of (13) shows that, apart from the logarithmic dependence on the average concentration, the behavior of $R(n)$, which decreases with growing average concentration of defects, is strongly affected by the exponent s in (7). For a rupture according to Griffith ($s = 1/2$), the rate of the $R(n)$ decrease is fairly small, viz., $R \simeq 0.1\,p_0$ only in the range of concentrations n close to n_{cr}, where, as noted earlier, solution (13) is inapplicable. The "liquid" model ($s = 3/2$) yields a substantially stronger dependence $R(n)$. In the range of substantial average defect concentrations, $R \simeq 10^{-2}p_0$, which is considerably closer to the experimentally observed rupture strength in real solids. The quantitative discrepancy between the theory and experiment should apparently be sought both in the insufficient accuracy of (7) for the condition of brittle fracture and in the fact that this model ignores the collective behavior and the interaction of defects, which are of special significance in a critical state. In this connection one should note that a similar consideration of the data from [2, 3] yields even less satisfactory conclusions on the concentration dependence of ultimate strength, because these papers give understated values for the probability of rupture since dangerous situations were not considered there.

SUPPLEMENT

Determination of the probability that, out of N points equiprobably distributed throughout the volume V, m points get into the volume Δ

Let us consider $G \subset R^3$, $\int_G d^3\mathbf{r} = V$ and $D \subset G$, $\int_D d^3\mathbf{r} = \Delta$. Let N points numbered from 1 to N be randomly distributed over G. Let us denote by W_m the probability for m points $(1 < m < N)$ to get inside D with an accuracy up to a parallel translation of the latter. We denote by $A_{i_1 \ldots i_m}$ the event of the points with numbers $i_1 \ldots i_m$ (the numbers are assumed to be different) getting inside D. Then, according to the formula of composition of probabilities we have

$$W_m = P(\bigcup A_{i_1 \ldots i_m}) = \sum P(A_{i_1 \ldots i_m}) - \sum P(A_{i_1 \ldots i_m} \bigcap A_{i'_1 \ldots i'_m}) + \sum P(A_{i_1 \ldots i_m} \bigcap A_{i'_1 \ldots i'_m} \bigcap A_{i''_1 \ldots i''_m}) - \ldots . \quad \text{(S.1)}$$

In the general case W_m, $P(A_{i_1 \ldots i_m})$, $P(A_{i_1 \ldots i_m} \bigcap A_{i'_1 \ldots i'_m})$ etc. depend on both V and Δ and on the form of the G and D sets. However, to a first approximation in Δ/V these probabilities do not depend on the form of D and G. This follows from the fact that the dependence on the form manifests itself only when the points, after getting inside D, lie near the boundary of G (in a layer with a thickness $\sim \Delta^{1/3}$). But the contribution of these configurations of the points to the total probability is a quantity of a higher order of smallness in Δ/V than the probability itself.

Let us demonstrate that only the first sum in (S.1) should be retained as a first approximation in Δ/V. The remaining terms are quantities of a higher order of smallness. Note that the event $A_{i_1 \ldots i_m i_{m+1}}$ consists in that the points $i_1 \ldots i_m$ get into D, while the point i_{m+1} is positioned so that all the points $i_1 \ldots i_m i_{m+1}$ may also get into the set D. But this is fulfilled only when the i_{m+1}th point lies near $i_1 \ldots i_m$ (at least no farther away than at a distance $\Delta^{1/3}$ from each of them). Hence, $P(A_{i_1 \ldots i_m i_{m+1}})$ is of a higher order of smallness than $P(A_{i_1 \ldots i_m})$. At the same time $P(A_{i_1 \ldots i_m} \bigcap A_{i'_1 \ldots i'_m} \bigcap \ldots) \simeq P(A_{j_1 \ldots j_s})$, where $\{j_1 \ldots j_s\} = \{i_1 \ldots i_m; i'_1 \ldots i'_m\}$ (another notation is introduced since some identical points may be found among $i_1 \ldots i_m; i'_1 \ldots i'_m$). In this case, $s > m$ and, hence, $P(A_{j_1 \ldots j_s}) = o(P(A_{i_1 \ldots i_m}))$ which proves the assumption made.

Thus, only the first sum of identical terms remains in (S.1):

$$W_m = C_N^m P(A_{1 \ldots m}) + o(P(A_{1 \ldots m})). \quad \text{(S.2)}$$

When calculating $P(A_{1 \ldots m})$ the form of G and D can be chosen arbitrarily, if we are concerned with only the first term in the expansion of $P(A_{1 \ldots m})$ into a power series of Δ/V. As G and D, let us take cubes with the sides L and λ, respectively.

To solve an auxiliary problem, we find the probability P_m for the points $1, \ldots, m$, randomly distributed over the segment $[0, L]$, to get inside a segment of length λ. Denote by $\widetilde{P}_m$ the probability that the points $1, \ldots, m$ get inside the segment λ in a definite order (say, in the order of increasing of the coordinate: $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_m$). Then

$$P_m = m! \widetilde{P}_m. \quad \text{(S.3)}$$

Let $(\widetilde{P}_m | x_1 \ldots x_k)$ be the conditional probability $\widetilde{P}$ on condition that the first point got into the segment $[x_1, x_1 + dx_1]$, the second into the segment $[x_2, x_2 + dx_2]$, and so on. Let us denote $\widetilde{P}_m$ via $(\widetilde{P}_m | x_1)$:

$$\widetilde{P}_m = \int_0^L (\widetilde{P}_m | x_1) \frac{dx_1}{L}; \quad \text{(S.4)}$$

$(\widetilde{P}_m | x_1)$ via $(\widetilde{P}_m | x_1, x_2)$; $(\widetilde{P}_m | x_1, x_2)$ via $(\widetilde{P}_m | x_1, x_2, x_s)$; and so on. For $(\widetilde{P}_m | x_1, x_2, x_3, \ldots, x_k)$ we obtain

$$(\widetilde{P}_m | x_1, \ldots, x_k) = \int_{x_k}^{x_1 + \lambda} (\widetilde{P}_m | x_1, \ldots, x_{k+1}) \frac{dx_{k+1}}{L}. \quad \text{(S.5)}$$

Obviously, for $x_m \in [x_{m-1}, x_1 + \lambda]$

$$(\widetilde{P}_m | x_1, \ldots, x_m) = 1. \tag{S.6}$$

Proceeding from (S.6) and (S.5), it can easily be shown by induction that

$$(\widetilde{P}_m | x_1, \ldots, x_n) = \frac{1}{(m-n)!}(x_1 + \lambda - x_n)^{m-n} \frac{1}{L^{m-n}}. \tag{S.7}$$

Indeed, for $n = m$ (S.7) coincides with (S.6), and integrating (S.7) for $n = k + 1$ yields

$$\int\limits_{x_k}^{x_1+\lambda} \frac{1}{(m-k-1)!}(x_1 + \lambda - x_{k+1})^{m-k-1} \frac{dx_{k+1}}{L} \frac{1}{L^{m-k-1}} = \frac{1}{(m-k)!}(x_1 + \lambda - x_k)^{m-k} \frac{1}{L^{m-k}},$$

i.e., condition (S.5) is fulfilled too. Let us substitute $(\widetilde{P}_m | x_1)$ from (S.7) into (S.4):

$$\widetilde{P}_m = \int\limits_0^L \frac{\lambda^{m-1}}{(m-1)!} \frac{1}{L^{m-1}} \frac{dx_1}{L} = \frac{1}{(m-1)!}\left(\frac{\lambda}{L}\right)^{m-1} \tag{S.8}$$

and, according to (S.3), we have

$$P_m = m\left(\frac{\lambda}{L}\right)^{m-1}. \tag{S.9}$$

It should be noted that in calculating P_m we neglected λ/L in relation to unity.

For a cube, $P(A_{1\ldots m})$ is just equal to P_m^3. Taking into consideration that $\lambda^3 = \Delta$ and $L^3 = V$, we arrive at

$$P(A_{1\ldots m}) = m^3\left(\frac{\Delta}{V}\right)^{m-1} + o\left(\left(\frac{\Delta}{V}\right)^{m-1}\right). \tag{S.10}$$

Substituting this into (S.2), we ultimately obtain

$$W_m = m^3 C_N^m \left(\frac{\Delta}{V}\right)^{m-1} + o\left(\left(\frac{\Delta}{V}\right)^{m-1}\right). \tag{S.11}$$

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