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# SPACE-TIME TOPOLOGY AS A CONSEQUENCE OF THE DYNAMICS OF CLOSED BOSON STRINGS. II. THE SECOND APPROXIMATION

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**The possibility of constructing a theory where the space-time topology would be determined by the dynamics of closed boson strings is analyzed. The analysis has the form of an expansion into a series with respect to the coupling constant. The second-order expansion is considered. Some new constraints on the admissible space-time topology are found in addition to those in the first-order approximation.**

In the previous paper [1] we described an approach in which the space-time topology was determined by the dynamics of strings, and also considered the zeroth- and first-order approximations with respect to the string coupling constant. The purpose of the present paper is to analyze the second-order approximation. In accordance with our approach, a cohomological complex will be constructed and a topological invariant will be found in the given approximation. It will be shown that the requirement of the differential nilpotency in the second-order approximation strengthens the constraints on the possible choice of the space-time topology.

1. To the second-order approximation there corresponds the one-loop diagram and also a diagram describing the four-string interaction. As before, the expression for the interaction vertex is taken in accordance with formula (1) in [1].

Let us find the system of equations implied by the property of nilpotency of the total differential.

Because in the first-order approximation the differential is expressed by formula (3′) in [1],

$$d^1\omega = d^0\omega + F^1(\omega) = d^0\omega + g\Phi * \omega,$$

we shall seek the total differential in the form

$$d\omega = d^0\omega + F^1(\omega) + F^2(\omega) + F^3(\omega) + \ldots. \tag{1}$$

where $F^i(\omega)$ is a linear functional on the set of differential forms $\omega$, which is proportional to the $i$th power of the coupling constant. By virtue of (1), we obtain the following expression for the square of the differential:

$$\begin{aligned}[d]^2\omega = {} & [d^0]^2\omega + \{d^0F^1(\omega) + F^1(d^0\omega)\} + F^1(F^1(\omega)) \\ & + \{d^0F^2(\omega) + F^2(d^0\omega)\} + F^2(F^1(\omega)) + F^1(F^2(\omega)) + \ldots.\end{aligned}$$

In the first approximation we have

$$d^0(F^1(\omega)) + F^1(d^0\omega) + F^1(F^1(\omega)) = (d^1\Phi) * \omega = 0,$$

i.e., the condition of nilpotency of the differential in the first-order approximation holds on the set of closed forms. This relation corresponds to the covariant equation for the total differential:

$$[d]^2\omega = (d\Phi) * \omega. \tag{2}$$

Using (2) we derive a system of linked equations for the functionals $\Phi$ and $F^i$:

$$\begin{aligned}
&[d^0]^2 = 0, \\
&d^0(F^i(\omega)) + F^i(d^0\omega) = F^{i\prime}(\omega), \\
&d^0(F^1(\omega)) + F^1(d^0\omega) + F^1(F^1(\omega)) = (d^1\Phi) * \omega, \\
&F^{2\prime}(\omega) + g(F^2(\Phi * \omega) + \Phi * F^2(\omega)) = gF^2(\Phi) * \omega,
\end{aligned} \tag{3}$$

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In particular, for the functional $F^2$ we obtain the system

$$\begin{cases} F^{2\prime}(\omega) = 0, \\ F^2(\Phi * \omega) + \Phi * F^2(\omega) = F^2(\Phi) * \omega, \end{cases} \tag{4}$$

which supplements conditions (1) and (4).

System (3) must be supplemented with a condition following from the property of the differential to raise the order of the form by unity:

$$|F^i(\omega)| = |\omega| + 1. \tag{5}$$

2. Our task is to find the form of the functionals $F^i$, which are solutions to the system of equations (4), (5).

To satisfy condition (5) it is necessary to introduce the dual differential $\overline{d}^0$ in the contour space, which does not raise but lowers the order of the form by unity. By definition, this differential operator satisfies the relation

$$d^0\overline{d}^0\omega = \omega, \qquad \overline{d}^0 d^0\widetilde{\omega} = \widetilde{\omega}. \tag{6}$$

Alongside the zeroth-order dual differential $\overline{d}^0$, there also exists the total dual differential $\overline{d}$:

$$\begin{cases} d\overline{d}\omega = \omega, \quad \overline{d}d\widetilde{\omega} = \widetilde{\omega}, \\ [\overline{d}]^2\omega = 0, \end{cases} \tag{7}$$

for which the conditions

$$\overline{d}\omega \equiv \overline{d}^0\omega + \overline{F}^1(\omega) + \overline{F}^2(\omega) + \cdots = \overline{d}^0\omega + \overline{\Phi} \circ \omega + \overline{F}^2(\omega) + \ldots$$

hold.

Note that conditions (6) and (7) are understood in the sense of cohomological equivalence classes, i.e., accurate to the addition of an arbitrary exact form to $\omega$.

Writing in full Eq. (7) we find a system of linked equations for the functionals $\overline{F}^i$, which is obtained by means of the transformation $F \to \overline{F}$, $\Phi \to \overline{\Phi}$, $* \to \circ$ in (3). Besides, there appears an additional system of equations:

$$\begin{cases} d^0(\overline{\Phi} \circ \omega) = -\Phi * \overline{d}^0\omega, \\ \overline{d}(\Phi * \omega) = -\overline{\Phi} \circ d^0\omega. \end{cases} \tag{8}$$

$$\begin{cases} \overline{\Phi} \circ (\Phi * \omega) + \overline{d}^0 F^2(\omega) + \overline{F}^2(d^0\omega) = 0, \\ \Phi * (\overline{\Phi} \circ \omega) + d^0\overline{F}^2(\omega) + F^2(\overline{d}^0\omega) = 0. \end{cases} \tag{9}$$

Equations (8) express additional operation rules on the functionals $\Phi$ and $\overline{\Phi}$, and system (9) supplements the system of equations (4) for the functionals $F^2$ and $\overline{F}^2$. Analyzing system (4), (7), (9), common for $F^2$ and $\overline{F}^2$, with consideration for rules (8) and rules (4) in [1], we find the following form of the desired functionals in the second-order approximation with respect to the coupling constant:

$$\begin{aligned} F^2 &= g^2(d^0[\overline{\Phi} \circ ((\overline{d}^0\Phi) * d^0\omega)] - \Phi * \overline{d}^0(\Phi * \omega)), \\ \overline{F}^2 &= g^2(\overline{d}^0[(\overline{d}^0\Phi) * d^0(\Phi \circ \omega)] - \overline{\Phi} \circ d^0(\overline{\Phi} \circ \omega)). \end{aligned} \tag{10}$$

The composition $*$ means the transition of a single string $z$ into two strings $x$ and $y$. Therefore the dual composition $\circ$ means the reverse transition $z' \to x, y$. The first terms in expressions (10) describe the one-loop interaction, and the second terms describe a four-string vertex.

3. We now find the topological invariant in the second-order approximation with respect to the coupling constant, which determines the effective action.

We first determine the integration measure on $U^2$. Since the first-order approximation with respect to the coupling constant means a higher order of contact as compared to the first-order approximation, the neighborhood over which the integration extends in the expressions for the topological invariants contains

$U^1$. Therefore in the integral measure on the neighborhood of $U^2$ there appear two new terms as compared to $U^1$:

$$\int\limits_2^a = \frac{1}{2}\Big\{ \int\limits_{-\tau_1}^{\tau_1} d\tau' d\tau'' \oint Dx(\sigma, \tau') \oint Dy(\sigma, \tau'')\delta(\tau' - \tau'')v(xyz)$$

$$= \int\limits_{\tau'}^{\tau_1} d\tau \oint Dz(\sigma, \tau)\overline{v}(zxy) \int\limits_{\tau}^{\tau_1} d\tilde{\tau} d\tilde{\tilde{\tau}} \oint Dx(\sigma, \tilde{\tau}) \oint Dy(\sigma, \tilde{\tilde{\tau}})$$

$$\int\limits_2^b = \frac{1}{2} \int\limits_{\tau_0}^{\tau_1} d\tau \oint Dz(\sigma, \tau)v(xyz) \int\limits_{\tau}^{\tau_1} d\tau' d\tau'' \oint Dx(\sigma, \tau')$$

$$= \oint Dy(\sigma, \tau'')\delta(\tau' - \tau'')\overline{v}(zxy) \int\limits_{\tau'}^{\tau_1} d\tilde{\tau} \oint Dz(\sigma, \tilde{\tau}).$$

The transformation rule for functionals in the transition from one neighborhood to another in the second-order approximation is written thus:

$$\delta^2\Phi = d^2\Psi = d^0\Psi + g\Phi * \Psi + F^2(\Psi). \tag{11}$$

As can easily be shown by means of direct verification, the topological invariant in the second-order approximation with respect to the coupling constant, i.e., an expression invariant relative to (11), takes the form

$$\Gamma^2 = \Gamma^1_{U^2} + \frac{2}{5}\int\limits_{U^2} \langle \Phi, F_a^2(\Phi, \overline{\Phi})\rangle + \frac{1}{2}\int\limits_{U^2} \langle \Phi, F_b^2(\Phi, \overline{\Phi})\rangle, \tag{12}$$

where

$$F_a^2 = g^2 d^0[\overline{\Phi} \circ ((\overline{d}^0\Phi) * d^0\Phi)],$$
$$F_b^2 = -g^2\Phi * \overline{d}^0(\Phi * \Phi).$$

Using (8) these expressions can be brought to the form not involving $\overline{\Phi}$:

$$F_a^2(\Phi) = g^2\Phi * \overline{d}^0[(\overline{d}^0\Phi) * d^0\Phi].$$

The variation of (12) results in equations of motion:

$$d^2\Phi = d^0\Phi + g\Phi * \Phi + F^2(\Phi) = 0.$$

One can see that the equations coincide with the closure condition for the form $\Phi$. Thus, the invariant $\Gamma^2$, which describes the topological properties of the contour space, assumes a descrete set of values on the set of closed forms.

The quantity $\Gamma^2$ is the one-loop approximation to the effective action, which immediately results in the second-quantized theory.

4. To illustrate the mechanism of the appearance of additional constraints on the space-time topology due to the second-order approximation, we consider a simple example of a topology admissible in the first-order approximation.

As an example we take a torus $T = R^{(D-1,1)}/l$, where $l$ is the lattice $\{e_i\}$ on the space $R^{(D-1,1)}$, $(e_i, e_j) \equiv \sum\limits_{IJ} e_i^I e_j^J \eta_{IJ} \equiv g_{ij}$. Consider expression (10) for $F_a^2$. From the physical standpoint, it describes the transition of a single string into two strings and the subsequent reverse transition of the two strings into the single string. If before the interaction the indices of the mapping of the string $z$ on the torus were zero, then after the interaction the indices of the mapping of the strings $x$, $y$ become equal to $\pm k_i \neq 0$, respectively,

on a lattice vector $e_i$. Upon repeated interaction all mapping indices vanish again. This interaction can be realized on a torus whose lattice contains at least two vectors $e_i$, $e_j$ ($\pi_1(T) \neq Z$, where $Z$ is the set of integers). Moreover, the dual operator $\circ$ involved in (8) is a composition $x, y \to \bar{z}$ on the dual lattice $\bar{l}$:

$$(\bar{e}_i, e_j) = \delta_{ij}.$$

This means that a lattice on the space-time must be self-dual.

In the approximation order with respect to the coupling constant we are dealing with, there appears one more constraint on the possible space-time topology in the case of a torus. Consider the graph of the lattice $l$, which means that the elements $e_i$ are denoted by points and the $i$th and $j$th elements are joined by a line if $(e_i, e_j) \neq 0$. Taking into account that we parametrized the contour space by passing consecutively from one lattice vector to another, from the requirement that the parametrization should be single-valued we conclude that any two points in the graph can be joined by only a single path. This is equivalent to the property that the graph possesses a tree structure, i.e., it contains no closed cycles.

Hence, taking account of the second-order approximation leads to additional constraints on the possible space-time topology as compared to the first-order approximation. It should be expected that the inclusion of higher-order approximations will result in a further concretization of the space-time topology and at the same time will elucidate some questions related to compactification.

## REFERENCES

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