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THE INVERSE PROBLEM FOR A ONE-DIMENSIONAL SINGULAR OSCILLATOR

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The inverse problem for the one-dimensional Schrödinger equation with a singular potential is solved using an oscillator with perturbation λx^{-2} as an example. The corrections to the potential were found to possess a weaker singularity as compared to the seed potential.

In this paper we consider the problem of calculating corrections to the potential of a one-dimensional singular oscillator ($\hbar = 2m = \omega/2 = 1$):

$$V(x) = U(x) + W(x) = x^2 + s(s+1)x^{-2}, \tag{1}$$

which appear upon a change in spectral data (energy levels and normalization constants). This problem arises as a natural completion of the earlier solved direct problem with the indicated potential [1–3] and the inverse problem for a one-dimensional harmonic oscillator [4] ($V(x) = x^2$, $-\infty < x < \infty$) and a radial oscillator [5] ($V(r) = r^2 + l(l+1)r^{-2}$, $0 < r < \infty$).

The inverse problem is an important part of any one-dimensional problem, and its solution for a potential with singularities at $x = 0$ and $x = \infty$ is by far not trivial. The general method for solving inverse problems with an increasing potential ($x \to \pm\infty$, $r \to \infty$) was found only as recently as the nineteen-eighties [4, 6, 7].

Potentials with the singularity λx^{-2} have physical applications because they model the transformations of the Universe at the initial stage of its development [8] and describe the spectroscopy of diatomic molecules [9]. Some other applications of these potentials are indicated in [10].

It is clear that a change in the experimental data on energy levels or in information about the time of the tunnel transition of the Universe to another state may result in a variation of the potential without changing the singularity at $x = 0$. Therefore not only theoretical but also applied problems require a solution of the inverse problem for the pivot potential with the singularity λx^{-2}. One of the methods for solving this inverse problem is outlined in the present paper. The method is applicable to the case of "any" smooth potential $U(x)$ in (1). It is presented in detail only for the case of the seed potential $U(x) = x^2$ in (1), for which there are closed analytical expressions for some corrections to the potential (exact solutions).

The stationary Schrödinger equation (SE) for this system possesses the well-known exact solutions (see [11], p. 158) and is thoroughly investigated (see, e. g., [12, 13]). However, in our opinion, the correct choice of even solutions to the SE with potential (1) was justified only recently [2, 3]. The even SE eigenfunctions with Hamiltonian (1) have the form ($x > 0$)

$$\psi_+(x) = x^{-s}\exp\{-x^2/2\}F\Big(\frac{1-2s-E}{4}, -s+1/2, x^2\Big), \tag{2}$$

where E is energy and $F(a, b, z)$ is the confluent hypergeometric function which is regular at $z = 0$ (see [14], p. 321). For $x \to +0$, function (2) is "normalized" by the condition

$$\lim_{x\to+0} x^s\psi_+(x) = 1. \tag{3}$$

The even energy levels are equidistant [12, 3]:

$$E_{n+} = -2s + 1 + 4n_+, \quad n_+ = 0, 1, 2, \ldots, \tag{4}$$

as well as the odd ones:

$$E_{n-} = 2s + 3 + 4n_-, \tag{5}$$

whose wave functions ($x > 0$)

$$\psi_-(x) = x^{s+1}\exp\{-x^2/2\}F\Big(\frac{3+2s-E}{4}, s+\frac{3}{2}, x^2\Big), \tag{6}$$

"normalized" by the condition

$$\lim_{x\to+0}(x^{-s}\psi_-(x))' = 1, \tag{7}$$

coincide with the radial wavefunctions for oscillator (1) ($0 \leqslant x < \infty$) (see [11], p. 140).

Using functions (2) and (6) one can construct pivot solutions to SE (1) and pass to the inverse problem of reconstructing the corrections to potential (1) following variation of the spectral data by the Gel'fand–Levitan (GL) method [15] as applied to the case of attractive potential [16]. To write down the GL integral equation it is necessary to construct the set of regular (pivot) solutions to SE (1) throughout the axis $-\infty < x < \infty$. For this purpose one must indicate a rule for the extension of any solution of the SE with the singular potential $W(x) = s(s+1)x^{-2}$ in (1) across the point $x = 0$ (not necessarily with an even smooth potential $U(x)$ in (1)). As such a rule we suggest the local even (odd) extension to $x < 0$ for the SE solution components satisfying the same boundary conditions for $x \to +0$ as the functions $\psi_+(x)$ and $\psi_-(x)$ (2), (6) in [1].

We choose $x = -\infty$ as a regular point. Then it is convenient to take as a regular solution to SE (1) the solution that has at $x \to -\infty$ the same asymptotic behavior as the SE solutions for the harmonic oscillator ($s = 0$) decreasing for $x \to -\infty$:

$$\psi_{as} = \exp\{-x^2/2\}|x|^{\frac{E-1}{2}}, \quad x \to -\infty. \tag{8}$$

This means that the regular solution $\psi_r(x)$ to SE (1) is subject to the "normalization" condition (an analog of conditions (3) and (7))

$$\lim_{x\to-\infty}\exp\{x^2/2\}|x|^{-\frac{E-1}{2}}\psi_r(x) = 1. \tag{9}$$

Such are the solutions to SE (1) ($x < 0$, and E is arbitrary)

$$\psi_r(x) = |x|^{-s}\exp\{-x^2/2\}U\Big(\frac{1-2s-E}{4}, -s+1/2, x^2\Big), \tag{10}$$

where $U(a, b, z)$ is the confluent hypergeometric function that is regular for $z \to +\infty$ (see [14], pp. 321 and 325).

On extending function (10) to $x > 0$ according to the above rule (with the change of the transition direction) and taking into account the relationship between the various solutions to SE (1) (see [14], p. 321) we find

$$\psi_r(x) = A_+\psi_+(x) + A_-\psi_-(x), \quad x > 0, \tag{11}$$

$$A_+ = \frac{\pi}{\cos\pi s\,\Gamma\big(\frac{3+2s-E}{4}\big)\Gamma\big(\frac{1}{2}-s\big)},$$
$$A_- = \frac{\pi}{\cos\pi s\,\Gamma\big(\frac{1-2s-E}{4}\big)\Gamma\big(\frac{3}{2}+s\big)}. \tag{12}$$

For all energies outside the spectrum (4), (5), the function $\psi_r(x)$ has an asymptotics increasing for $x \to +\infty$:

$$\psi_r(x) = \frac{2\pi}{\cos\pi s\,\Gamma\big(\frac{3+2s-E}{4}\big)\Gamma\big(\frac{1-2s-E}{4}\big)}x^{-\frac{E+1}{2}}\exp\{x^2/2\}. \tag{13}$$

On the spectrum (4), (5), the regular solutions have the form

$$\psi_{r+} = (-1)^{n_+}n_+!L_{n_+}^{-\left(\frac{1}{2}+s\right)}(x^2)|x|^{-s}\exp\{-x^2/2\}, \tag{14}$$

$$\psi_{r-} = (-1)^{n_-}n_-!L_{n_-}^{\left(\frac{1}{2}+s\right)}(x^2)|x|^{s}\exp\{-x^2/2\}, \tag{15}$$

where $L^{\alpha}_{m}(z)$ is the generalized Laguerre polynomial (see [14], p. 580). The asymptotics of $\psi_{r\pm}(x)$ for $x \to +\infty$ is of course decreasing:

$$\psi_{r\pm}(x) = \exp\{-x^2/2\}x^{\frac{E-1}{2}}. \tag{16}$$

In what follows it is convenient to index the wave functions and the energy levels in the order of increasing energy (the even and odd levels alternate). In this case we have ($n_+ = m/2$, $m = 0, 2, 4 \ldots$; $n_- = (m-1)/2$, m=1,3,5 ...)

$$E_m = (2m+1) - 2(-1)^m s, \quad m = 0, 1, 2 \ldots. \tag{17}$$

The normalization constants C_m of regular functions (14) and (15) have the form

$$C_m = \Big[\int\limits_{-\infty}^{\infty} \psi_m^2\, dx\Big]^{-1} = \Big[\Gamma\Big(\frac{E_m - 2s - 1}{4}\Big)\Gamma\Big(\frac{E_m + 2s + 3}{4}\Big)\Big]^{-1}. \tag{18}$$

Using the pivot regular solutions (10), (11), (14), and (15) one can write down the GL integral equation [15, 16] for a preassigned change in the spectral data (E_m, C_m). The solution to the GL equation allows one to find corrections ΔV to the potential and new regular (for the potential $V + \Delta V$) wave functions $\overline{\psi}_r(x)$ (on the spectrum of $\overline{\psi}_m(x)$). In the case of a change in the finite number of spectral data, the GL equation is solved in quadratures. The expressions for ΔV and $\overline{\psi}_m(x)$ are given in [16] (where the lower limit of integration 0 should be replaced by $-\infty$ in all the formulas).

Of interest are the asymptotics of $\overline{\psi}_m$ and of ΔV for $x \to 0, \pm\infty$. To find them, use is made of asymptotics (8), (13), and (16) and formulas (2) and (6) for $x \to +0$.

We describe the behavior of $\overline{\psi}_m$ and ΔV in some typical cases.

1. A change in the normalization constant: $C_m \to \overline{C}_m = C_m + \Delta C_m$. In this case

$$\overline{\psi}_m(x) = \exp\{-x^2/2\}|x|^{(E_m-1)/2} = \psi_m(x), \quad x \to -\infty, \tag{19}$$

$$\overline{\psi}_m(x) = \psi_m(x)C_m/\overline{C}_m, \quad x \to +\infty, \tag{20}$$

$$\overline{\psi}_n(x) = \psi_n(x), \quad x \to \pm\infty. \tag{21}$$

$$\overline{\psi}(x) \sim |x|^{-s}, \quad x \to 0. \tag{22}$$

The parity is not retained after C_m has been changed:

$$\overline{\psi}_{n\pm}(-x) \neq \pm\overline{\psi}_{n\pm}(x), \tag{23}$$

$$\Delta V = -4\Delta C_m \exp\{-x^2/2\}|x|^E, \quad x \to \pm\infty, \tag{24}$$

$$\Delta V \sim |x|^{1+2s}\varepsilon(x), \quad x \to 0, \quad m = 1, 3, 5 \ldots, \tag{25}$$

where $\varepsilon(x)$ is the signature function:

$$\Delta V = \frac{4\Delta C_m A_+^2 s\varepsilon(x)|x|^{-(1+2s)}}{1 + (1/2)\Delta C_m C_m^{-1}} + \frac{2(\Delta C_m)^2 A_+^4 |x|^{-4s}}{(1 + (1/2)\Delta C_m C_m^{-1})^2}, \quad x \to 0, \quad m = 0, 2, 4 \ldots, \tag{26}$$

$$A_- = \frac{(-1)^{m/2}\Gamma(m/2 + 1/2 - s)}{\Gamma(1/2 - s)}. \tag{27}$$

The first term has the strongest singularity in correction (26).

2. The inclusion of an additional level $\overline{E}$ with normalization constant $\overline{C}$ yields

$$\Delta V = \begin{cases} -4\overline{C}\exp\{-x^2\}|x|^E, & x \to -\infty, \\ \dfrac{4\overline{C}A_+^2(\overline{E})s\varepsilon(x)|x|^{-(1+2s)}}{1 + \overline{C}\int\limits_{-\infty}^{0} \psi_r^2(\overline{E}, x)\, dx}, & x \to 0, \\ -4, & x \to +\infty. \end{cases} \tag{28}$$

where $A_+(\overline{E})$ is determined by formula (12); for $\psi_r(\overline{E}, x)$ see (10).

3. The exclusion of the level E_m leads to results close to (28), in which, however,

$$\Delta V = 4, \quad x \to +\infty. \tag{29}$$

4. A shift of the level gives

$$\Delta V \sim x^{-2}, \quad x \to +\infty. \tag{30}$$

Note that, in contrast to the radial case ($0 \leqslant r < \infty$), all the results of the inverse problem throughout the axis x ($-\infty < x < \infty$) have a trivial twofold degeneration dependent on the choice of the regular point ($-\infty$ or $+\infty$). The formulas obtained for $s \to 0$ and $x \to \pm\infty$ coincide with the analogous formulas for the one-dimensional oscillator ($s = 0$) partly presented in [4] and for the radial oscillator ($x \to \infty$) in [5].

As to the asymptotics of ΔV for $x \to 0$ described by (26) and (28), which is also characteristic of cases 3 and 4, it is a new nontrivial result. Note that the singularity in (28) ($-1/2 < s < 1/2$) is weaker than that of the pivot potential (1) (and of the related point potential ($\delta V = -2s\delta(x)|x|^{-1}$ [1]).

In the case when the even functions of Hamiltonian (1) are for some reason chosen in the form of a combination of $\psi_+(x)$ and $\psi_-(x)$ in (2) and (6) (this possibility was discussed in [1]), the proposed method of solving the inverse problem can be easily generalized.

However, if the singular part of the potential has a weaker singularity than (1), e.g., $W = \lambda|x|^{-\nu}$, $0 < \nu < 2$ (see [17]), then the choice of an even SE solution requires additional physical reasons as compared to [2, 3]. Therefore the solution to the inverse problem presented here cannot be extended automatically to the values of ν less than 2. This case will be investigated separately.

The solution to the inverse problem found in this paper can also be used for some other potentials with the singularity λx^{-2}.

In this case the solution to the inverse problem can be applied for solving the Korteweg–de Vries equation with singular initial data ($V \sim \lambda x^{-2}$, $x \to 0$; $V \to 0$, $x \to \pm\infty$) by analogy with the nonsingular Korteweg–de Vries problem [15].

Thus, we have managed to fill one of the few gaps in the classical problem of reconstructing the potential from spectral data and revealed, in so doing, the nontrivial behavior of ΔV for $x \to 0$.

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