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CHARACTERISTICS OF INTERMODE STRICTION COUPLING IN SPHERICAL ELECTROACOUSTIC RESONATORS

G. V. Belokopytov, N. P. Pushechkin, and V. N. Semenenko

Integrated characteristics of the intermode striction interaction of electromagnetic and acoustic oscillations in spherical resonators have been studied. Selection rules are established and integrated striction interaction characteristics are calculated with consideration for the vector character of the fields and the electrostriction tensor anisotropy. There was good agreement between the calculated and experimental data on striction parametric excitation of radial elastic oscillations by UHF pumping in potassium tantalate dielectric resonators.

The investigation of wave interaction in nonlinear resonators is readily reduced (by means of expansion with respect to natural oscillations) to the problem of coupled nonlinear oscillators. Then the efficiency of the nonlinear coupling between different modes is characterized by integrated coefficients whose magnitude depends on the spatial distribution of interacting fields and the nonlinear properties of the medium. This approach was applied in an analysis of parametric interactions in optical [1] and microwave [2, 3] band resonators and also of the phenomena of magnetoacoustic resonance [4] and striction parametric excitation [5] in ferroelectrics. Upon a suitable normalization all parametric generation characteristics can be presented as functions of the oscillatory system pumping level, tuning, and quality factor. This enabled the authors of the above-mentioned papers to investigate the nonlinear oscillation dynamics without detailed information about the spatial structure of the interacting modes.

However, it is the specific features of the spatial distribution of fields (electromagnetic, elastic, and magnetostatic) that determine, in terms of integrated coefficients, the interaction efficiency and, accordingly, the normalization scale in equations for coupled modes, the parametric generation threshold, and, ultimately, the very fact of oscillation excitation on certain modes. The general expressions for the integrated coefficients are well known but their calculation is rather difficult, and therefore until recently one had to confine oneself to either analysis of simplest one-dimensional models [1, 2] or crude estimates based on similarity considerations [2, 5]. The only particular exception was the calculation of the integrated coefficient for the interaction due to the cubic nonlinearity on the fundamental mode of a spherical dielectric resonator carried out in [3] with consideration for the anisotropy of the nonlinear dielectric susceptibility.

The present paper deals with the calculation and experimental determination of integrated coefficients of the intermode striction coupling in spherical dielectric resonators. Here a unique feature is the fact that for an isotropic spherical resonator an analytical representation of the fields of natural electromagnetic and acoustic oscillations is known [6]. This makes it possible to simplify expressions for integrated coefficients, formulate selection rules for effectively interacting mode combinations, and create a rational calculation program.

The integrated intermode coupling coefficients for striction interaction are introduced in the following way. The electric induction **D** and the mechanical displacement vector **U** in resonators are represented as expansions with respect to the natural standing waves which correspond to the normal frequencies ω_a and Ω_w and form orthogonal families of modes normalized by means of the relations

$$\int \varepsilon^{-1}\mathbf{D}^{(a)}\mathbf{D}^{(b)}\,dV = \delta_{ab}, \quad \Omega_r^2 \int \rho \mathbf{U}^{(r)}\mathbf{U}^{(s)}\,dV = \delta_{rs}, \tag{1}$$

where V is the dielectric resonator volume, ε is the permittivity, and ρ is the crystal density. Substitution of the expansions $\mathbf{D}(\mathbf{r},t) = A_a(t)\mathbf{D}^{(a)}(\mathbf{r})$ and $\mathbf{U}(\mathbf{r},t) = B_w(t)\mathbf{U}^{(w)}(\mathbf{r})$ into the electro- and elastodynamics equations with regard for the material equations and the orthogonality relations (1) results in the following system of equations [5] for the amplitudes of coupled electromagnetic and acoustic oscillations:

$$\begin{aligned} \ddot{A}_f + \omega_f^2 A_f &= -R_{fa}\dot{A}_a - I^*_{far}A_aB_r - F_f^{el}(t), \\ \ddot{B}_w + \Omega_w^2 B_w &= -\theta_{wr}\dot{B}_r - (1/2)I_{wab}A_aA_b - F_w^m(t). \end{aligned} \tag{2}$$

Here the subscripts a, b, f and w, r index the electromagnetic and acoustic modes, respectively, and I^*_{far} and I_{wab} are the integrated intermode coupling coefficients in question:

$$I^*_{far} = \omega_f^2\left(K_{raf} - \frac{c^2}{\mu}S^*_{raf}\right), \quad I_{wab} = \Omega_w^2(K_{wab} - S_{wab}), \tag{3}$$

where

$$K_{wab} = \int_V G_{jinm}u_{ji}^{(w)}D_n^{(a)}D_m^{(a)}\,dV,$$

$$S_{wab} = \oint_S G_{jinm}U_i^{(w)}X_jD_n^{(a)}D_m^{(b)}\,dS,$$

$$S^*_{wab} = \oint_S([\varkappa\mathbf{D}^{(a)};\mathrm{rot}\,\mathbf{D}'] - [\mathbf{D}';\mathrm{rot}\,\varkappa\mathbf{D}^{(a)}])\,d\mathbf{S},$$

$\varkappa = 4\pi\varepsilon^{-1}$, $D'_n = G_{ijmn}u_{ij}^{(w)}D_m^{(b)}$, $u_{ij} = \partial U_i/\partial X_j$ are the components of the strain tensor, G is the electrostriction tensor, and X_i are the Cartesian coordinates. The volume integral K_{wab} is proportional to the electrostriction interaction energy in the dielectric resonator. The integrals S_{wab} and S^*_{wab} extend over the resonator surface (the vector $d\mathbf{S}$ is oriented along the outer normal). They describe the energy transformation effect of the electromagnetic wave upon its reflection from the vibrating resonator surface.

The electromagnetic modes of a spherical dielectric resonator are divided into two classes—E and H. Furthermore, the acoustic oscillations of an isotropic elastic ball are classified into purely transverse torsional (T) and mixed (S) oscillations. In all the indicated cases the spatial distribution of the electric and elastic displacement vectors can be expressed in terms of the vector spherical functions $\mathbf{L}$, $\mathbf{M}$, and $\mathbf{N}$:

$$\mathbf{L} = \mathrm{grad}\,\psi; \quad \mathbf{M} = \mathrm{rot}(\mathbf{r}\psi); \quad \mathbf{N} = (1/k)\,\mathrm{rot}\,\mathbf{M}, \tag{4}$$

where $\mathbf{r}(r,\theta,\varphi)$ is the unit vector along the radius and ψ is a solution to the scalar wave equation

$$\Delta\psi + k^2\psi = 0 \tag{5}$$

in spherical coordinates. For instance, we have

$$\mathbf{E} = A_E\mathbf{N}, \quad \mathbf{E} = A_H\mathbf{M} \tag{6 a}$$

for the electromagnetic E and H modes and

$$\mathbf{U} = A_T\mathbf{M}, \quad \mathbf{U} = A_s\mathbf{L} + B_s\mathbf{N} \tag{6 b}$$

for the acoustic T and S modes, respectively.

The solutions to (5) are known [6] to have the following form:

$$\psi(r,\theta,\varphi) = (kr)^{-1/2}J_{n+1/2}(kr)P_n^{(m)}(\cos\theta)\begin{cases}\sin m\varphi, \\ \cos m\varphi,\end{cases} \tag{7}$$

where the constants A and B in (6) and the wave number k are determined by the boundary conditions and normalization (1). All the indicated types of modes are identified by a triad of indices: radial p, zonal n, and azimuthal m. The index p determines the number of half-waves that fit within the ball radius, and n

and m determine the spatial angular distribution of fields by formula (7). For elastic radial oscillations we have $m = n = 0$ (the S_{p00} type modes).

To identify uniquely the version of three-mode striction interaction and the corresponding integrated coefficients I_{wab} and I^*_{wab} for a spherical dielectric resonator one should indicate the class to which each of the three standing waves in (3) belongs and to set 9 indices: (p_a, n_a, m_a) for the acoustic mode and (p_1, n_1, m_1) and (p_2, n_2, m_2) for the electromagnetic modes.

The number of possible interaction versions is immense, and therefore one of the main problems is the selection of combinations of effectively interacting modes. The natural frequencies of these modes must satisfy the frequency synchronism and spatial overlap conditions. The first condition means that the difference of the electromagnetic mode frequencies must be equal to the acoustic mode frequency with an accuracy of the order of magnitude of the electromagnetic resonance band width. Because of the very large distinction in the electromagnetic and elastic wave velocities the synchronism condition for the modes of a spherical dielectric resonator can actually be satisfied in the following two cases [5]. 1. If the electromagnetic mode frequencies are nearly degenerate and the elastic mode frequency is approximately equal to their difference, then a three-mode interaction takes place. In a spherical resonator each of the electromagnetic modes is $(2m + 1)$-fold degenerate. A small perturbation of the indicated oscillations removes the degeneration and makes the three-mode interaction possible. Moreover, for $\varepsilon \gg 1$ the $H_{l,n,m}$ and $E_{l,n-1,m'}$ modes are nearly degenerate. 2. If the electromagnetic resonance band width is comparable with the acoustic frequency or exceeds it, then the pumping and combination frequency excitation takes place on the same electromagnetic mode. Such a two-mode or a "mode preserving" regime can also be energetically effective.

The spatial overlap condition means that the coefficients I_{wab} and I^*_{wab} must not vanish. We consider the consequences of this requirement for the most important case when the interaction involves electromagnetic modes of one class with the same indices p and n. Analysis of coefficients (3) with consideration for (4), (6), and (7) makes it possible to establish the following selection rules. In the striction interaction only those elastic S modes, including radial ones, can take part for which the zonal indices are even, $n_a = 2l$ (where l is an integer), and the azimuthal indices satisfy the relation

$$|m_a \pm m_1 \pm m_2| = 0, 2, 4. \tag{8}$$

The shear modes interact with the electromagnetic ones only if their indices are odd: $n_a = 2l + 1$. In this case the azimuthal indices must also satisfy relations (8). For $m_1 = m_2$ (a two-mode excitation regime) we have $m_a = 4j$.

When calculating integrated coefficients (3) for a spherical resonator, the volume and surface integrals are divided into finite sums of the products of one-dimensional integrals with respect to spherical coordinates. Here an analogy is observed with the problem of determining the Clebsch–Gordan coefficients [7] where one also deals with integration of triple products of spherical harmonics with respect to the angular coordinates. However, this analogy cannot be used because of the high complexity of the problem of integrated coefficients (owing to the vector character of the wave fields, the electrostriction anisotropy in crystals, and the necessity of integrating with respect to the radius). In particular, this complexity manifests itself in the fact that selection rules (8) do not coincide with the addition rules for angular momenta.

Having in mind the possibility of comparison with experiment, specific calculations were carried out for resonators made of virtual potassium tantalate ferroelectric having cubic symmetry. In the calculation of electromagnetic oscillations we took $\varepsilon = 4\,000$, which corresponds to helium temperatures. For an approximate description of elastic oscillations of a $KTaO_3$ ball we used the model of an isotropic elastic medium with the following effective stiffness constants: $c_{11} = 3.81 \times 10^{12}$ dyn cm^{-2}, $c_{44} = 1.25 \times 10^{12}$ dyn cm^{-2}, and $\rho = 6.93$ g cm^{-3}. For this choice of the constants the velocities of the transverse and longitudinal acoustic waves are equal to the mean values determined for real acoustically anisotropic $KTaO_3$ crystals in measurements in the crystallographic directions [100], [110], and [111] (see [8]).

The electrostriction tensor in a $KTaO_3$ crystal has three independent components. According to the experimental data in [9], $G_{11} = -7$, $G_{12} = 0.4$, and $G_{44} = -0.4$ in SGS units. So the anisotropy of striction properties is considerable, and it must be explicitly taken into account when calculating the integrated coefficients.

According to [5], for the threshold power of striction parametric excitation we have the formula

$$P_{\text{th}} = \omega_f F/(Q_1 Q_2 Q_a K^2_{ef}), \tag{9}$$

where ω_f is the cyclic frequency of microwave pumping; F is the tuning function of the system; Q_1 and Q_2 are the quality factors of the electromagnetic and combination frequency oscillation modes (in a two-mode regime we have $Q_1 = Q_2$); Q_a is the quality factor of acoustic oscillations; and $K_{ef}^2 = I_{wab}I_{wab}^*/\omega_f^2\Omega_a^2$ are the intermode coupling coefficients.

The software package described in [10] was used to calculate K_{ef}^2 in a $KTaO_3$ spherical resonator for approximately 500 two- and three-mode interaction versions with excitation of symmetric radial and purely shear oscillations. The dependence of the coefficient K_{ef}^2 on the elastic constants is particularly simple for them so that we can set

$$K_{ef}^2 = \frac{\Phi\varepsilon^2}{c_{ef}V}, \tag{10}$$

where $c_{ef} = c_{11}$ for radial oscillations and $c_{ef} = c_{44}$ for shear oscillations, and the quantity Φ depends only on the electrostriction coefficients and the spatial distribution of the fields of the interacting modes.

Information about the efficiency of the nonlinear striction coupling between the modes can be obtained in experiment by measuring the threshold parametric excitation power. In the same measurement cycle it is possible to determine parameters characterizing the tuning of the oscillatory system, the dielectric resonator coupling coefficient and the quality factor, and to find Φ using (9) and (10).

A batch of spherical resonators of potassium tantalate crystals and also of alloyed $KTaO_3$:Li crystals (with replacement of potassium by lithium up to 10% in the initial mixture) was prepared for the experiments. Measurements were performed at a temperature of 4.2 K in the $(\omega_f/2\pi)$ range of 8.2–10 GHz. The connection between the spherical dielectric resonator and the microwave section was achieved by means of a short-circuit loop at the end of a coaxial cable, and in this case only magnetic oscillations (the H_{pnm} modes) were excited.

The resonators of the batch under investigation showed striction excitation of about 20 different acoustic modes in the $(\Omega_a/2\pi)$ interval from 2 to 30 MHz. Note that in striction excitation of homogeneous traveling waves (e.g., during stimulated scattering of Mandelstam–Brillouin) the Bragg conditions imply the constraint $\Omega_a/\omega_f < 2(v_a/v_{el})$, where v_a and v_{el} are the sound and light velocities in the medium. In our experiments this limit was exceeded because the standing waves in a dielectric resonator are sure to be inhomogeneous.

We managed to observe parametric processes in different (two- and three-mode) regimes involving different mode combinations by varying the tuning conditions and changing the pumping frequency. Excitation of radial acoustic oscillations (the S_{p00} modes) occurred only in the two-mode regime. The selection rules do not forbid excitation of radial oscillations in the three-mode regime, but, according to the calculation results, in this case the value of Φ is four orders of magnitude lower than for the mode preservation regime.

Shear oscillations (the T_{pnm} modes) were more effectively excited in the three-mode regime, though there was also a two-mode excitation, including the excitation of the modes T_{12m} forbidden by the selection rules. To explain this phenomenon one should bear in mind that the selection rules and the integrated coefficients were determined with neglect of the anisotropy of the acoustic properties of $KTaO_3$. The difference in transverse wave velocities depending on the direction and polarization attains 20%, which creates difficulties in the identification of acoustic oscillations and leads to large differences between the calculated and measured values of Φ.

The results of the determination of striction interaction parameters for the most reliably identified modes are presented in Table 1. Rather good agreement between the calculated values of the coefficients Φ and the experiment results on the excitation of radial oscillations was observed. For the three-mode interaction with the T modes there was sometimes quantitative agreement, too, between the experimental and calculated values of K_{ef}^2. For instance, the calculation showed that in the interaction between the H type oscillations and the torsional T_{11} mode the integrated coefficients vanished. On the other hand, no striction excitation of the T_{11} mode occurred in any conditions of the experiment. At the same time, Table 1 shows that, as a whole, the accuracy and the degree of agreement between the results for the shear modes are lower than for radial oscillations. Analysis and additional calculations [10] make it possible to indicate the reason for this discrepancy. The distribution of strain fields of the T modes is such that their striction coupling with electromagnetic oscillations is much weaker than in the case of radial modes. For example, the maximum values of the parameter Φ for radial and shear acoustic oscillations differ by two orders of magnitude.

This result depends weakly on the character of the electrostriction anisotropy and indicates that the nonlinear interactions in different parts of the resonator cancel out one another to a considerable extent. In these conditions their efficiency is strongly affected by the wave field perturbations produced by the

Table 1

Characteristics of Nonlinear Spherical Electroacoustic Resonators

Acoustic mode (number of specimens)	$\Omega_a/2\pi$, calculation*, experiment	Electromagnetic modes	Φ	
			Experiment	Calculation
S_{100} (11)	6.13 5.96 ± 0.09	H_{11m}	4.0 ± 0.4	4.4 ± 0.4
		H_{21m}	6.4 ± 0.6	6.4 ± 0.4
		H_{31m}	6.5 ± 0.6	6.8 ± 0.4
		H_{41m}	7.4 ± 0.7	6.9 ± 0.4
S_{200} (3)	14.26 14.35 ± 0.07	H_{11m}	3.9 ± 0.4	3.1 ± 0.3
		H_{21m}	5.0 ± 0.5	5.6 ± 0.3
		H_{31m}	7.1 ± 0.7	6.7 ± 0.4
T_{13m} (8)	5.20	H_{111}, $H_{111'}$	0.10–0.25	0.22
	5.37 ± 0.14	H_{211}, $H_{211'}$	0.19	0.12
T_{23m}^{**} (9)	11.36	H_{111}, $H_{111'}$	0.005	0.0067
	10.55–11.70	H_{211}, $H_{211'}$	0.01–0.09	0.07 ± 0.01

*Reduced to the diameter of 1 mm.
**Close to degeneration with the S_{16m} mode.

anisotropy of the elastic properties and random imperfections of the resonator, and this is responsible for the discrepancy between theory and experiment.

Although in the three-mode regime the striction coupling integrated coefficients were much smaller than in the two-mode case, the more favorable tuning conditions [5] provided a threshold reduction and the excitation of acoustic oscillations at a low pumping level (down to several microwatts).

A study of the alloying effect on the striction parametric interaction efficiency in $KTaO_3$ confirmed the relation $P_{\text{th}} \sim \varepsilon^{-7/2}$ implied by (9) and (10) if we bear in mind that at a fixed frequency ω_f the volume of the dielectric resonator excited on the given mode is proportional to $\varepsilon^{-3/2}$. At the same time, the acoustic frequencies and the values of Φ in 5%-alloyed and unalloyed crystals did not in fact differ, which indicates that the replacement of potassium ions by lithium affects weakly the elastic stiffness and electrostriction constants.

The investigation performed has demonstrated the possibility of calculating the integrated characteristics of nonlinear spherical electroacoustic resonators with consideration for the vector character of the interacting fields and the anisotropy of the material medium. The calculation accuracy achieved makes it possible not only to obtain a qualitative explanation of the experimentally observed striction excitation properties but also to attain quantitative agreement. This opens prospects for predicting the threshold of striction parametric excitation for other materials and frequency bands.

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11 December 1991

Department of Oscillation Physics