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THE EFFECT OF VARIATION OF THE REFRACTIVE INDEX AND THICKNESS OF LAYERS IN THIN-LAYER INTERFERENCE FILTERS ON THEIR OPTICAL PARAMETERS

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Changes in the refractive index of a medium to be matched and in the matching wavelength upon variation of the refractive indices and thicknesses of layers in thin-layer interference filters (TIF) were studied theoretically (the optical thickness of the layers is smaller than a quarter wavelength) for TIF with two and three layers in a period. New properties of such filters were discovered, in particular, the invariance of these changes with respect to the number of periods when the refractive indices of the layers vary. Simple analytical expressions were obtained that allow an adequate determination of the shift in the parameters under consideration. A computer-aided numerical analysis confirmed the theoretical predictions.

A feasibility of designing matching structures (TIF) showing new optical and structural properties was reported previously [1, 2]. The new features of these structures mainly are (i) the invariance of their amplitude characteristics and the overall optical thickness with respect to the number of layers and (ii) the possibility of synthesizing such structures with layer thicknesses substantially smaller than a quarter wavelength. It was demonstrated in [3–6] that a number of problems could be solved within various wavelength ranges by using such thin-layer interference matchers with two or three layers (of the same optical thickness) in a period.

We shall consider the effect on the optical parameters of the TIF produced by small deviations in the refractive indices of the layers from the initially specified ones, these deviations being assumed uniform over the entire cross section of the TIF. We shall also assume that the wave is normally incident on the plane of the TIF layers and is not restricted in any way in the plane normal to the direction of its propagation, that there is no absorption in the layers, and $dn_i/d\lambda = 0$ (dispersion is neglected).

It has been shown in [3] that all possible solutions to the problem of finding the layer thicknesses of an arbitrary layered structure that would provide complete transmission of the propagating wave may be obtained by using the system of independent equations

$$M_{21} = M_{12} n_s n_L, \tag{1}$$

$$M_{11} = M_{22} n_s / n_L, \tag{2}$$

where M_{ij} are the elements of the characteristic matrix, n_L is the refractive index of the medium where the wave propagates, n_s is the refractive index of the medium to be made translucent, or, which is the same, the matching ability of the TIF;

$$M_{11} = {}^N m_{11} U_{K-1}(X) - U_{K-2}(X), \qquad M_{12} = {}^N m_{12} U_{K-2}(X),$$
$$M_{22} = {}^N m_{22} U_{K-1}(X) - U_{K-2}(X), \qquad M_{21} = {}^N m_{21} U_{K-2}(X),$$

where ${}^N m_{ij}$ are the elements of the characteristic matrix for one N-layer period of the structure and $U_k(x)$ are the second-order Chebyshev polynomials,

$$x = \frac{1}{2}({}^N m_{11} + {}^N m_{22}).$$

For two- and three-layer TIF, by substituting the M_{ij} matrix elements in an explicit form into system (1)–(2) we obtain, respectively,

$$\begin{cases} n_{s0} = \dfrac{1-(n_{20}/n_{10})T_{10}T_{20}}{1-(n_{10}/n_{20})T_{10}T_{20}} n_L, \\ n_{s0} = \dfrac{\sum_{i=1}^{3} n_{i0}T_{i0}}{\sum_{i=1}^{3} \frac{1}{n_{i0}}T_{i0}} \dfrac{1}{n_L}, \end{cases} \quad \text{and} \quad \begin{cases} n_{s0} = \dfrac{\sum_{i=1}^{3} n_{i0}T_{i0} - (n_{10}n_{30}/n_{20}) \prod_{i=1}^{3} T_{i0}}{\sum_{i=1}^{3} \frac{T_{i0}}{n_{i0}} - \frac{n_{20}}{n_{10}n_{30}} \prod_{i=1}^{3} T_{i0}} \cdot \dfrac{1}{n_L}, \\ n_{s0} = \dfrac{1-(n_{30}/n_{10})T_{10}T_{30}-(n_{30}/n_{20})T_{20}T_{30}-(n_{20}/n_{10})T_{10}T_{20}}{1-(n_{10}/n_{30})T_{10}T_{30}-(n_{20}/n_{30})T_{20}T_{30}-(n_{10}/n_{20})T_{10}T_{20}} n_L. \end{cases}$$

Here $T_i \equiv \tan(2\pi n_i d_i/\lambda)$; n_{s0} is the refractive index for the medium to be made translucent at given (constant) n_{i0} ($i = 1, 2, 3$), n_{10}, n_{20}, and n_{30}, which are the refractive indices of, respectively, the layer adjacent to the medium to be made translucent and of the subsequent layers.

Let n_i be given by the relation $n_i = n_{i0} + \Delta n_i$, and the condition $\Delta n_i/n_i \equiv \delta$, $i = 1, 2, 3$ is fulfilled for the two- and the three-layer TIF, respectively. Solving the obtained system to a first approximation with respect to the TIF matching ability and the wavelength of matching, for the two-layer TIF we obtain:

$$\frac{\Delta\lambda}{\lambda} = \delta\Big(1 + \frac{2n_{s0}}{F-L}\Big), \tag{3}$$

$$n_s = n_{s0}\Big[1 + 2\delta\Big(1 - \frac{F}{F-L}\Big)\Big], \tag{4}$$

where

$$F = B_1\Big(\frac{c_1 n_{10} + c_2 n_{20}}{n_L} - \Big(\frac{c_1}{n_{10}} + \frac{c_2}{n_{20}}\Big)n_{s0}\Big),$$
$$L = B_2(c_1T_{10} + c_2T_{20})\Big(n_{s0}\frac{n_{10}}{n_{20}} - n_L\frac{n_{20}}{n_{10}}\Big),$$
$$B_1 = \Big(\frac{T_{10}}{n_{10}} + \frac{T_{20}}{n_{20}}\Big)^{-1}, \qquad B_2 = \Big(1 - \frac{n_{10}}{n_{20}}T_{10}T_{20}\Big)^{-1},$$
$$c_i = (1+T_{i0})^2 \tan^{-1} T_{i0}.$$

In the particular case of equal optical thicknesses ($c_i \equiv c$, $i = 1, 2, 3$), expression (4) is simplified:

$$n_{s0} = n_{s0}(1 + 2\delta) \tag{5}$$

and a change in wavelength is given by:

$$\frac{\Delta\lambda}{\lambda} = \delta\Big(1 - \frac{B_2}{T_0 c(n_{10}/n_{20} - n_L^2/n_{10}^2)}\Big). \tag{6}$$

For a three-layer TIF and equal optical thicknesses, we obtain:

$$\Delta n_s = 2\delta\frac{C}{2A}\Big(1 \pm \frac{B + C - 2AD/C}{[(B+C)^2 - 4AD]^{1/2}}\Big), \tag{7}$$

where

$$A = SM - \frac{n_{20}}{n_{10}n_{30}}, \qquad \Sigma = \sum_{i=1}^{3} n_{i0}, \qquad S = \sum_{i=1}^{3} \frac{1}{n_{i0}},$$
$$B = \Big(SN - \frac{n_{20}}{n_{10}n_{30}}\Big)n_L, \qquad C = \Big(M\Sigma - \frac{n_{10}n_{30}}{n_{20}}\Big)\frac{1}{n_L}, \tag{8}$$
$$D = N\Sigma - \frac{n_{10}n_{30}}{n_{20}}, \qquad M = \frac{n_{10}}{n_{20}} + \frac{n_{10}+n_{20}}{n_{30}}, \qquad N = \frac{n_{30}}{n_{20}} + \frac{n_{20}+n_{30}}{n_{10}},$$

$$T^2 = T_0^2 \frac{1 + 2\delta f(n_{i0}, n_{s0})/(n_{s0} - n_L)}{1 + 2\delta f(n_{i0}, n_{s0})/(n_{s0}M - n_L N)}, \tag{9}$$

where $i = 1, 2, 3$; and $f(n_{i0}, n_{s0} = \frac{[(n_{s0}/n_L)M - N]\Sigma - (n_{10}n_{30}/n_{20})(n_{s0}/n_L - 1)}{2n_{s0}A - B - C}$.

It was shown in [6] that a specific feature of thin-layer structures with three layers in a period is the possibility of making translucent two media with different refractive indices at two different wavelengths simultaneously. The longer wavelength of matching will be termed the main peak of matching and the shorter one the additional peak of matching. The plus sign in (7) corresponds to the main peak of matching and the minus sign to the additional one. In particular, it may be readily shown that when the relationship between the refractive indices of the layers

$$\frac{n_{20}}{n_{10}n_{30}}\Sigma = \frac{n_{10}n_{30}}{n_{20}}S \tag{10}$$

holds true and their relative changes are the same, the refractive index of the medium that undergoes matching at the additional peak does not change.

In the case of periodic structures with two layers in a period, we use the relationship [3, 5, 8] between the layer thickness and the number of periods

$$d_2 = \frac{\lambda}{2\pi n_2}\tan^{-1}[K(P + Q(K-1))]^{-1/2}$$

and obtain the following simple relationships for the wavelength change and the new value of the medium matching ability:

$$\frac{\Delta\lambda}{\lambda} = \delta, \tag{11}$$

$$n_s = n_{s0}(1 + 2\delta). \tag{12}$$

Similar expressions are also obtained for a three-layer TIF by using the relationship [6] between T^2 and the number of periods:

$$T^2 = \frac{1}{[(K+1)^2 - 4] + (1/2)K(K-1)(N+M)} \quad \text{for} \quad K = 2p+1, \quad p = 0, 1, 2, \ldots,$$

$$T^2 = \frac{1}{[(K+1)^2 - 3] + (1/2)K(K-1)(N+M)} \quad \text{for} \quad K = 2p, \quad p = 1, 2, \ldots.$$

The accuracy of the analytical expressions obtained was checked by a computer-aided numerical experiment for wavelengths in the medium IR range ($\lambda_0 = 12$ μm) and for the refractive indices of the TIF layers from 2 to 4 (most typical of this spectral range) that were varied no more than within 10%. The data of the numerical experiment showed a good agreement (accuracy better than 1% for the main peak of matching and 4% for the additional peak) with the theoretical predictions.

Consider now the effect of layer thickness variation on the characteristics of a three-layer TIF. Since the layer thickness variation leads directly only to changes in T, we introduce a TIF of unequal layer thicknesses, having defined T_j as follows:

$$T_j = T + \Delta_j.$$

Then, using system of equations (1)–(2), we obtain the equation of coupling in the general form:

$$\frac{1 \pm H\chi_{ij}}{1 \pm P\chi_{ji}} = \frac{1 \mp V\psi_{ij}}{1 \mp W\psi_{ji}}, \qquad i, j = 1, 2, 3, \tag{13}$$

where

$$\chi_{ij} = \sum_i \Delta_i n_i - q\Big(T^2 \sum_i \Delta_i + T_i \sum_{i \neq j}\sum \Pi_{ij} + \prod_i \Delta_i\Big),$$

$$\chi_{ji} = \sum_i \frac{\Delta_i}{n_i} - q\Big(T^2 \sum_i \Delta_i + T \sum_{i \neq j}\sum \Pi_{ij} + \prod_i \Delta_i\Big),$$

$$\psi_{ij} = n_{21}(T\Sigma_{21} + \Pi_{21}) + n_{31}(T\Sigma_{31} + \Pi_{31}) + n_{32}(T\Sigma_{32} + \Pi_{32}),$$

$$\psi_{ji} = n_{12}(T\Sigma_{12} + \Pi_{12}) + n_{13}(T\Sigma_{13} + \Pi_{13}) + n_{23}(T\Sigma_{23} + \Pi_{23}),$$

$$H = \Big[T\Big(\sum_i n_i - \frac{1}{q}T^2\Big)\Big]^{-1}, \qquad P = \Big[T\Big(\sum_i \frac{1}{n_i} - qT^2\Big)\Big]^{-1},$$

$$V = \big[1 - T^2(n_{21} + n_{31} + n_{32})\big]^{-1}, \qquad W = \big[1 - T^2(n_{12} + n_{13} + n_{23})\big]^{-1},$$

$$n_{ij} = n_i/n_j, \qquad \Pi_{ij} = \Delta_i\Delta_j, \qquad \Sigma_{ij} = \Delta_i + \Delta_j, \qquad q = \frac{n_2}{n_1 n_3}.$$

Now let us analyze the equation of coupling for small variation of thickness of two layers:

$$\Delta_2 = 0, \qquad \Delta_i = k\Delta_3 \ll T$$

and of three layers:

$$\Delta_1 = k_1\Delta_3, \qquad \Delta_2 = k_2\Delta_3, \qquad \Delta_j \ll T.$$

For the former case, we find that the coupling coefficient k is a constant that is determined by T and n_i:

$$k = k_0(T, n_i),$$

and thus we come to a linear dependence between the varied thicknesses

$$\Delta d_1 = \frac{n_3}{n_1}\Delta d_3 k_0, \tag{14}$$

as well as between changes of the refractive index of the medium to be made translucent and the thickness variation of one of the layers:

$$\Delta_3 = \Delta n_s \frac{1}{\gamma n_{s0}}, \tag{15}$$

$$\gamma = k_0(H\xi_1 - P\xi_2) + H\zeta_1 - P\zeta_2,$$

$$\xi_1 = n_1 - \frac{1}{q}T^2, \qquad \xi_2 = \frac{1}{n_1} - qT^2, \qquad \zeta_1 = n_3 - \frac{1}{q}T^2, \qquad \zeta_2 = \frac{1}{n_3} - qT^2.$$

As is seen from the equation of coupling and the relationship for k, the value of k_0 changes in going to the additional matching peak. In the case of the additional peak λ_p, when the TIF layer thickness varies the wavelength must be changed too to fulfill the conditions of matching. For this change $\Delta\lambda$ in the first of the two cases under consideration we obtain

$$\frac{\Delta\lambda}{\lambda} = \frac{\Delta D_3}{\Delta D}\frac{k_0' - k_0}{k_0' - 1} \equiv \frac{\Delta D_1}{D_1}\frac{(k_0'/k_0) - 1}{k_0' - 1}, \tag{16}$$

where $D = d_i n_i$; k_0 and k_0' are found from the expressions for k at given values of T, T_P, and n_i.

To determine the refractive index of the medium that can be matched at the additional peak, we may use relationship (15) replacing λ and λ_p, etc.

When the thickness of all the three layers is varied, the equation of coupling has the form

$$k_1 = Q - k_2 R, \tag{17}$$

where

$$Q = \frac{Vt_3 + Pt_2 - Ht_1 - Wt_4}{Hq_1 + Wq_4 - Pq_2 - Vq_3}, \qquad R = \frac{Hr_1 + Wr_4 - Pr_2 - Vr_3}{Hq_1 + Wq_4 - Pq_2 - Vq_3},$$

$$q_1 = n_1 - \frac{1}{q}T^2, \qquad r_1 = n_2 - \frac{1}{q}T^2, \qquad t_1 = n_3 - \frac{1}{q}T^2,$$

$$q_2 = \frac{1}{n_1} - qT^2, \qquad r_2 = \frac{1}{n_2} - qT^2, \qquad t_2 = \frac{1}{n_3} - qT^2,$$

$$q_3 = n_{21} + n_{31}, \qquad r_3 = n_{21} + n_{32}, \qquad t_3 = n_{31} + n_{32},$$

$$q_4 = n_{12} + n_{13}, \qquad r_4 = n_{12} + n_{23}, \qquad t_4 = n_{13} + n_{23},$$

i.e., the explicit dependence upon Δ_j is absent. Now the choice of k_1 and k_2 is governed by the requirements to the refractive index of the medium to be matched and to the wavelength of the additional peak, for which the following dependencies on the layer thickness variation could be readily obtained:

$$\frac{\Delta n_s}{n_{s0}} = \Delta_3(k_2 X + Y), \tag{18}$$

$$\frac{\Delta\lambda_p}{\lambda_p} = \frac{1}{D}\,\frac{Q'\Delta D_3 - (\Delta D_1 + \Delta D_2)}{Q' - R' - 1}, \tag{19}$$

where

$$X = Q(Hq_1 - Pq_2) + Ht_1 - Pt_2,$$

$$Y = H(r_1 - q_1 R) + P(r_2 - q_2 R).$$

Therefore, we obtained the relationships between the changes in layer thickness of a three-layer TIF and the refractive index of the medium that can be matched; these relationships uniquely determine the shift in the wavelength of the additional matching peak if the wavelength of the main peak is kept unchanged. Moreover, one notices some specific features of the behavior of the matched medium upon variation of the layer thicknessess. First of all, in both cases there is a linear dependence between Δ_i and Δ_j $(i, j = 1, 2, 3)$ and a corresponding dependence between Δ_i and n_s; when two layers are varied at a given Δ_i, these dependencies are preset strictly; when all the three layer thicknesses are varied, we have two variables and a third one (Δ_k or n_s) that is rigorously determined by the coupling condition obtained.

When the changes of all the layer thicknesses of the TIF under study are not large and there is a certain relationship between the thickness variations:

$$\begin{cases} \Delta D_2 X = -\Delta D_3 Y, \\ \Delta D_1 = \Delta D_3\left(Q + \dfrac{Y}{X}R\right), \end{cases} \tag{20}$$

a compensation of the changes in refractive index of the medium to be matched will take place.

A variation of the layer thicknesses at a constant wavelength of the main matching peak is always accompanied by a change of the wavelength of the additional peak, which can be found from relationships (16), (19).

The numerical experiment on a computer showed that the divergence between the thus calculated and the theoretically predicted values did not exceed 1.6% when the optical thicknesses of the layers were varied by no more than 20%. In calculating the dependence of $\Delta_{i,j}$ and n_s on the variation of the optical thickness Δ_k not exceeding 40%, the divergence between the calculated and theoretically predicted values was no more than 10% for $\Delta_{i,j}$, and no more than 6% for n_s.

In particular, when all the three layer thicknesses are varied so that $\Delta_i = \Delta_j \neq \Delta_k$ (i.e., one varies the thickness of one of the layers and adjusts the other two in the same way), a calculation can be performed for layer optical thicknesses using more exact formulas that give more accurate results. In this case the variation of the optical thicknesses is given by:

$$T_k = aT, \qquad T_i = T_j \equiv T.$$

Substituting these $T_{i,j,k}$ into system (1)–(2) and solving it with respect to T and n_s at the given a and $n_{1,2,3}$, we obtain the following relationships:

$$T^2 = \{an_Lq - (a/n_Lq) \mp [(an_Lq - a/n_Lq)^2 - 4\Gamma\Phi]^{1/2}\}/2\Gamma, \tag{21}$$

$$T^2 = \frac{Z \mp [Z^2 - 4(SM - aq)(\Sigma N - a/q)]^{1/2}}{2(SM - aq)}, \tag{22}$$

where

$$\Gamma = Nn_Laq - \frac{Ma}{n_Lq},$$

$$\Phi = Mn_LS - \frac{M\Sigma}{n_L},$$

$$Z = n_L(SN - aq) + \frac{1}{n_L}\left(\Sigma M - \frac{a}{q}\right), \qquad q = \frac{n_2}{n_1n_3}.$$

The quantities S, Σ, M, and N are determined by relationships (8). From (21) and (22), we have two pairs of solutions for T and n_s. At $a = 1$, they correspond to matching at the main (−) and additional (+) peaks of wavelengths. Now if we need to match two media at $a = 1$ by means of one structure while retaining unchanged, e. g., the wavelength of the main peak, then, to obtain the shift of the additional peak and the refractive index of the medium to be adjusted at this peak, we shall calculate the layer thicknesses for the given a from (21) and the loading to be matched at the main peak from (22). Then, under the condition of small thickness variation, from (19) we shall find the new wavelength of the additional peak λ_p, and from the initial condition

$$T_k = aT, \quad \text{where} \quad T \equiv \tan\frac{2mn_id_i}{\lambda_p} = \tan\frac{2\pi n_jd_j}{\lambda_p},$$

we shall find the value of a^* and of n_{sc} (from (22)) that correspond to the additional peak of matching at a constant wavelength of the main peak and a variation of layer thicknesses corresponding to the given a. In such a case, the layer thicknesses and the refractive index n_{sf} of the medium matched at the main peak are calculated by exact formulas, and the accuracy of matching reaches 10^{-4}. As before, the error in the λ_p (and, accordingly, n_{sc}) determination does not exceed 1.6%.

The following conclusions may be drawn from the above analysis. (i) When the refractive indices of the layers in a two- or a three-layer TIF are varied, there exists a linear dependence of the wavelength of matching and the refractive index of the medium to be made translucent upon the characteristics varied. By using the derived relations (7), (9), (11), and (12), the characteristics required for the matching can be found. (ii) With a large number of periods, changes in the wavelength and refractive index of the medium to be matched do not depend on the number of periods. (iii) In the case of a three-layer TIF, when relation (10) between the refractive indices of the layers is fulfilled, during their variation one can change the refractive index of the medium being matched at the main peak while leaving constant the refractive index of the medium being matched at the additional peak. (iv) Similarly, when the TIF layer thicknesses are varied, there is a linear relationship between Δ_i and Δ_j (i, j = 1, 2, 3) and, respectively, n_s and Δ_i. If the thickness varies for all the three TIF layers, we have two variables and a third one (Δ_k or n_s) that is rigidly determined by the coupling condition obtained. (v) When the thickness variation is small for all the three layers in a three-layer TIF and when condition (20) is fulfilled for one of the peaks of matching, the refractive index of the medium being matched at this wavelength will remain unchanged. (vi) In all the cases, when the layer thicknesses are varied at a constant wavelength of the main peak of matching, the wavelength of the additional peak changes and can be found from relations (16) and (19). Moreover, the relations derived enable one to determine changes in wavelengths and refractive indices required for proper matching of media when the refractive indices and layer thicknesses are varied in TIF with two or three layers in a period.

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