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# Z PARAMETERS OF A PIEZOELECTRIC TRANSDUCER FOR SOLID-STATE GRAVITATIONAL ANTENNAS

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**A system of $Z$ coefficients is computed for a piezoelectric transducer as a minor shift generator of a Weber-type solid-state gravitational antenna.**

## 1. INTRODUCTION

The most general analysis of dynamic and noise characteristics of solid-state gravitational antennas [1] with a piezoelectric transducer can be made on the basis of the theory of linear quadrupole transducers [2, 3]. A piezoelectric transducer as a linear quadrupole device with $Z$ parameters is described by the following system of equations [4]:

$$F_\mu = \frac{k^D}{p} v_\mu + \frac{h}{p} I, \qquad V = \frac{h}{p} v_\mu + \frac{1}{pc^\varepsilon} I. \tag{1}$$

Here, $F_\mu$, $v_\mu$ are the mechanical force and velocity, respectively; $V$, $I$ are the voltage and current, respectively; $p = d/dt$; $k^D = E^D s_0/r$, $c^\varepsilon = \varepsilon^\varepsilon s_0/r$ are the mechanical rigidity and capacitance of the transducer, respectively; $E^D$ and $\varepsilon^\varepsilon$ are the modulus of elasticity and the permittivity of the piezoelectric material measured at constant electric flux and constant strain; $s_0$ and $r$ are the area and the thickness of the piezoelectric plate, respectively; and $h$ is the conversion factor.

Primary noise parameters of the transducer [1] caused by loss in the piezoelectric material have been determined in [5] on the basis of the fluctuation-dissipation theorem [6]: $k^D = k^D(p)$, $c^\varepsilon = c^\varepsilon(p)$.

The objective of this work is to calculate the system of $Z$ parameters of a piezoelectric transducer as a shift sensor of the solid-state gravitational antenna. The simplest antenna model is a homogeneous rod of length $L$ having at the middle a rigidly connected piezoelectric element. The effect of the transducer on the antenna is taken into account by introducing "volume" forces $P_{01,2}(t)\delta(x - x_{01,2})$, where $x_{01,2} = (L \pm r)/2$.

## 2. DERIVATION OF BASIC RELATIONS

Let $x_0$ be the point to which a concentrated force $P_0(t)\delta(x - x_0)$, $0 < x_0 < L$ is applied. Then, for the spectrum of longitudinal vibrations $\dot u(x, j\omega)$ of the elastic rod we obtain [7]

$$\left(\frac{d^2}{dx^2} + k^2\right)\dot U(x, j\omega) = -E^{-1}\dot P_0(j\omega)\delta(x - x_0) \tag{2}$$

with the boundary conditions

$$E\frac{d\dot u(x, j\omega)}{dx}\bigg|_{x=0,L} = \dot P_{1,2}(j\omega).$$

The following designations are used here: $k = \omega/v_s$, $v_s = (E/\rho)^{1/2}$ is sound velocity; $E$ is the modulus of elasticity; $\rho$ is density; $\dot P_{1,2}(j\omega)$ is the spectrum of external forces.

A general solution of inhomogeneous Eq. (2) can be given by

$$\dot U = A\exp\{jkx\} + B\exp\{-jkx\} - (kE)^{-1}\dot P_0 \sin k(x - x_0)V(x - x_0), \tag{3}$$

where $V(x)$ is a symmetric function.

The unknown constants $A(j\omega)$ and $B(j\omega)$ will be found from the boundary conditions:

$$A,\, B = -(2kE\sin kL)^{-1}[\dot P_2 + \dot P_0 \cos k(L - x_0) - \dot P_1 \exp\{\mp jkL\}]. \tag{4}$$

Substitution of (4) into (3) gives

$$\dot{U} = -(kE\sin kL)^{-1}[\dot{P}_2\cos kx - \dot{P}_1\cos k(L-x) + p_0(\cos k(L-x_0)\cos kx + \sin kL\cdot\sin k(x-x_0)\cdot V(x-x_0))], \qquad 0 < x_0 < L. \tag{5}$$

As $kE = \omega^2 M(kLS)^{-1}$, where $M = \rho LS$ is the mass of the rod and $S$ is the area of its end faces, we find from (5) the relative shift $\Delta\dot{U}(r, j\omega) = \dot{U}(x_2, j\omega) - \dot{U}(x_1, j\omega)$ with $x_{1,2} = (1/2)L \pm r$:

$$\Delta\dot{U} = kLS\Big(M\omega^2\cos\frac{kL}{2}\Big)^{-1}\Big[(\dot{P}_1 + \dot{P}_2 + \dot{P}_0\cos k(L-x_0))\sin\frac{kr}{2} - \dot{P}_0\cos\frac{kL}{2}\Big(V\Big(\frac{L+r}{2} - x_0\Big)\sin k\Big(\frac{L+r}{2} - x_0\Big) - V\Big(\frac{L-r}{2} - x_0\Big)\sin k\Big(\frac{L-r}{2} - x_0\Big)\Big)\Big]. \tag{6}$$

The absolute frequency $\omega_0$ of the fundamental mode is obtained from the equation

$$\cos\frac{k_0L}{2} = \cos\frac{\omega_0 L}{2} = 0, \qquad k_0L = \omega_0L/v_s = \pi. \tag{7}$$

Putting in (6) $\omega = \omega_0 - \Omega$, $|\Omega| \ll \omega_0$ one can write, in view of (7),

$$\Delta_0\dot{U} \approx 2S(M\omega_0\Omega)^{-1}[\dot{P}_1 + \dot{P}_2 + \dot{P}_0\cos k_0(L-x_0)]\sin\frac{k_0r}{2}. \tag{8}$$

Formula (8) can be readily generalized for the case of several concentrated forces:

$$\Delta_0\dot{U} \approx 2S(M\omega_0\Omega)^{-1}\Big[\dot{P}_1 + \dot{P}_2 + \sum_i \dot{P}_{0i}\cos k_0(L - x_{0i})\Big]\sin\frac{k_0r}{2}. \tag{9}$$

Now we can find the following expression from (9), with $i = 1, 2$ and $x_{01,2} = x_{1,2} = (1/2)(L \mp r)$:

$$\Delta_0\dot{U} \approx [2(M/4)\omega_0\Omega]^{-1}\Big(F_s - F_0\sin\frac{k_0r}{2}\Big)\sin\frac{k_0r}{2}, \tag{10}$$

where $F_s = (P_1 + P_2)S$, $F_0 = (P_{01} - P_{02})S = F_\mu(S/S_0)$.

Conversion of (10) $\Delta_0\dot{U}(r, j\omega) \leftrightarrow \Delta_0U(r, t)$ yields the equation

$$M_e(p + \omega_0^2/p)(p\Delta_0U) \approx (F_s - a(S/S_\mu)F_\mu)a, \tag{11}$$

where $M_e = (M/4)$ is the equivalent mass, $a = k_0r(2L)^{-1} = (\pi/2)(r/L) \ll 1$.

Note that for gravitational radiation [1] $F_s = F_h = -(8/\pi^2)M_e\omega_0^2h(t)L$, where $h(t)$ is the metric variation.

## 3. CALCULATION OF THE SYSTEM OF Z PARAMETERS OF THE PIEZOELECTRIC TRANSDUCER FOR A SOLID-STATE GRAVITATIONAL ANTENNA

Let $F = a(S/S_0)F_M$, $v = a^{-1}p(\Delta_0U)$. Then, we find from (11), (1), with $v_\mu = p(\Delta_0u) = av$:

$$F = a^2(S/S_0)k^Dp^{-1}v + a(S/S_0)hp^{-1}I,$$
$$V = ahp^{-1}v + (pc^\varepsilon)^{-1}I. \tag{12}$$

System (12) describes an electromechanical transducer based on the piezoelectric effect as a linear quadrupole transducer whose $Z$ parameters are

$$Z_{11}(p) = a^2(S/S_0)k^Dp^{-1}; \qquad Z_{12}(p) = a(S/S_0)hp^{-1};$$
$$Z_{21}(p) = ahp^{-1}; \qquad Z_{22}(p) = (pc^\varepsilon)^{-1}. \tag{13}$$

The input resistance $Z_{11}(p)$ of unidirectional transducer (12) running idle can be transformed as follows:

$$Z_{11}(p) = \beta M_e \omega_0^2/p, \qquad \beta = (r/L)(E^D/E). \tag{14}$$

An important characteristic of the antenna with a piezoelectric transducer is the output impedance $Z_{\text{out}}(p)$ [3]:

$$Z_{\text{out}}(p) = Z_{22}(p) - \frac{Z_{12}(p)Z_{21}(p)}{Z_c(p) + Z_{11}(p)}, \tag{15}$$

where $Z_c(p) = M_e(p + 2\delta_e + \omega_0^2/p)$ is the impedance of the mechanical system, and $\delta_e$ is the attenuation of the fundamental mode.

Let $K_{e\mu}$ be the electromechanical coupling coefficient, $K_{e\mu} = h(c^\varepsilon/k^D)^{1/2}$ [4]; then $Z_{12}(p)Z_{21}(p) = (h/p)^2a^2(s/s_0) = K_{e\mu}^2k^D(pc^\varepsilon)^{-1}Z_{11}(p)$. Therefore, in view of (14) and (15) we find

$$Z_{\text{out}}(p) \approx \frac{1}{pc^\varepsilon}\left[1 - \frac{K_{e\mu}^2\beta\omega_0^2}{(p^2 + 2\delta_e p + \omega_1^2)}\right],$$

where $\omega_1 = \omega_0(1 + \beta/2)$.

## 4. EXAMPLE

The GAISH solid-state gravitational antenna at the P. K. Shternberg State Institute of Astronomy has the following parameters: $L \approx 1.5$ m, $\rho \approx 2.7 \times 10^3$ kg/m$^3$, $S \approx 0.3$ m$^2$, $E \approx 67$ GPa, $v_s \approx 5080$ m/s, whence we find

$$M_e = (L\rho S)/4 \approx 304 \text{ kg}, \qquad f_0 = u_s/2L = 1693 \text{ Hz}.$$

The parameters of piezoelectric materials of the lead titanite-zirconate group are as follows: $K_{e\mu} \approx 0.4$; $d \approx 200 \times 10^{-12}$ C/H is the piezoelectrical modulus, $\varepsilon \approx 1.1 \times 10^3$ is the permittivity. The geometrical dimensions of the piezoelectric pellet are: $S_0 \approx 10^{-3}$ m$^2$, $r \approx 3 \times 10^{-2}$ m. The modulus of elasticity of a piezoelectric material, $E_D$, can be found from the formula [5]

$$E^D = K_{e\mu}^2(1 - K_{e\mu}^2)^{-1}\varepsilon_0\varepsilon d^{-2} \approx 43 \text{ GPa}$$

($\varepsilon_0 = 8.85 \times 10^{-12}$ is the permittivity).

Thus we have $\beta \approx 1.28 \times 10^{-2}$, $a \approx 3 \times 10^{-2}$, $S/S_0 \approx 3 \times 10^2$, $h \approx d^{-1}K_{e\mu}^2(1 - K_{e\mu}^2)^{-1} \approx 10^8$ V/m, $c_0^\varepsilon \approx 15$ pF.

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