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THE DYNAMICS OF CIRCLE MAPPING UNDER A PARAMETRIC PERTURBATION

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A standard sine mapping of a circle which effectively describes the transition from quasi-periodic motion to chaos in nonlinear systems is considered. For some parameter values corresponding to the chaotic dynamics of this mapping invariant distributions are calculated. It is shown that certain parametric actions on the mapping of a circle being in a chaotic mode result in suppression of chaos, that is, in parametric destochastization.

1. INTRODUCTION

When analyzing various physical systems one often resorts to constructing appropriate mathematical models. The modeling is meant to help the investigation and interpretation of the observed phenomena. Certainly, no mathematical model is completely adequate to the real object because its construction is usually based on various approximations and assumptions. For instance, this nearly always results in an inexact setting of necessary coefficients in equations. If a system is chaotic, these approximations may prove to be important and have an unexpectedly strong effect on the dynamics, so that the behavior of the model would be *qualitatively* incompatible with the original. In particular, in computer simulation, owing to the finite representation of a number, it may turn out that the model demonstrates regular properties, whereas the original shows chaos, or vice versa [1].

The aim of the present paper is to show that certain small additive and even purely parametric perturbations introduced in a system with chaotic behavior may result in *destochastization*, i. e., suppression of chaos. Earlier this possibility was illustrated with some examples [2–7]. However, it seems interesting to study the phenomenon of destochastization for sufficiently general systems (mappings) which effectively describe a definite scenario of chaos development [8].

The phenomenon of destochastization appears to be particularly important from the following two viewpoints. First, there arises a possibility of predicting deterministically the evolution of systems demonstrating a chaotic behavior. Second, the suppression of chaos by the parametric method, as distinguished from a force action [9–12], requires much less energy expenditures, which can be used for creating efficient neural networks [13–15].

2. DYNAMIC SYSTEM UNDER A PARAMETRIC PERTURBATION

In mathematical modeling, one usually constructs a dynamic system with continuous or discrete time. The object of study is differential equations (flows) in the first case and mappings (cascades) in the second case. Transitions from flows to cascades are rather well known and are achieved by constructing the Poincaré mapping. The essence of this transition is as follows. In the phase space of the dynamic system we choose a secant hypersurface S of codimension 1 transversal to the trajectories of motion. The trajectories intersect the hypersurface generically. Consider only those intersection points that appear as the trajectories pierce the hypersurface S only in one direction. Then the study of system's behavior reduces to analyzing the resulting intersection points. On introducing local coordinates on S one sees that these intersections can be regarded as consecutive transformations of a region on S into itself:

$$\mathbf{y}_{n+1} = \mathbf{f}(\mathbf{y}_n, \mu), \tag{1}$$

where $\mathbf{y}_n = \{y_n^1, \ldots, y_n^k\}$ are the coordinates of the nth point of intersection of the phase trajectory with the hypersurface S, n is the index of the intersection, $\mathbf{f} = \{f_1, \ldots, f_k\}$ is the law of point-to-point transformation on the surface S, and μ is a parameter.

There also exists a reverse construction enabling one to pass from mappings to flows [16].

As a mathematical model, a one-dimensional mapping is often considered, i.e., mapping (1) with $k = 1$:

$$x_{n+1} = f(x_n, \mu). \tag{2}$$

The description based on mapping (2) is also used when the dissipation in the system is sufficiently strong. In this case the points of intersection of the phase trajectory with the secant plane S fall on a nearly one-dimensional curve, and it is reasonable to describe such a curve by one-dimensional mappings. One-dimensional mappings also naturally appear regardless of differential equations. For example, the population size dynamics [17–19], some nonlinear systems with periodic excitation [20, 21], and a number of other systems [22, 23] are well described by Eq. (2).

Every dynamic system is almost always subject to the action of various factors. These factors can be the environment, other systems, etc. The external action can be additive or multiplicative. In the first case, when using mapping (2), one must add to the right-hand side a definite quantity ξ_n corresponding to the external action:

$$x_{n+1} = f(x_n, \mu) + \xi_n. \tag{3}$$

If the function f involves the parameter μ additively, such action can sometimes be regarded as a variation of the additive parameter depending on n: $\mu = \mu(n) \equiv \mu_n$. In this case mapping (3) is written as

$$x_{n+1} = F(x_n, \mu_n), \tag{4a}$$

where, we stress, the parameter μ_n is involved in the function F ***additively***.

In the second case the system is often described by means of equations or mappings with variable coefficients:

$$x_{n+1} = f(x_n, \mu_n), \quad \mu_n \equiv \mu(n), \tag{4b}$$

where the parameter μ_n is no longer involved in the function f ***additively*** but ***multiplicatively***. It is clear that mappings (4a) and (4b) are irreducible to each other.

An important application of mappings (3) and (4a, b) arises in the consideration of a k-dimensional ($k \geqslant 1$) network of one-dimensional mappings (2). When the interaction (coupling) in the network proceeds parametrically, then for some problems the dynamics of such a system is described in the approximation of a single mapping (4a, b) from the network and the constantly acting parametric background formed by the other mappings. In the case of coupling produced in a general way, mapping (3) or a combination of (3) and (4a, b) is considered.

The variation of the parameter μ_n from one iteration to another in mappings (4a) and (4b) can be interpreted as the action of the environment. If this action is nonperiodic, then, generally, μ_n also varies nonperiodically. When the action is periodic, the sequence of values of μ_n is formed by a set of subsequences of a length m: $\mu_{n+m} = \mu_n$, $\mu_{n+k} \neq \mu_n$, $1 \leqslant k < m$.

Below we consider only periodic variations of μ_n. What can the dynamics of mappings (4) be in this case? The answer to this question depends on the values the parameters $\mu_1, \ldots, \mu_m$ take on. If each of them corresponds to a regular behavior of mapping (2), then it should be expected that mappings (4a, b) will also manifest a regular dynamics; at least this is unambiguously suggested by the numerical studies.

Now let $\mu_i \in [\mu', \mu'']$, $i = 1, \ldots, m$, and let mapping (2) be in a chaotic evolution mode everywhere on the interval $[\mu', \mu'']$. This would seem to imply that mappings (4a) and (4b) should also demonstrate chaotic properties and have positive Lyapunov exponents. However, in the general case this is not so. It is well known [2–7] that upon a parametric action on a system of differential equations with a quasi-stochastic attractor the latter degenerates into a stable limit cycle. The parameter undergoing perturbation always remains in the region corresponding to the existence of such an attractor [2, 3]. A similar result turns out to be true for mappings. For definite parameter values (we denote them μ_i^d, $i = 1, \ldots, m$; the superscript "d" denotes "destochastization") satisfying the condition $\mu_i^d \in [\mu', \mu'']$, mappings (4a) and (4b) demonstrate a regular behavior.

To show this we consider a concrete mapping of a circle, which is often used when describing a transition of a system from quasi-periodic to chaotic oscillations.

3. PARAMETRIC PERTURBATIONS OF A CIRCLE MAPPING

In many cases the dynamics of a system describing a certain process can be represented as motion over an invariant torus with eigen frequencies ω_1 and ω_2. If the frequencies are commensurable, i. e., $\omega_1/\omega_2 = k/m$, where k and m are coprime numbers, then the phase trajectory closes up after k revolutions with respect to the coordinate φ and after m revolutions with respect to the coordinate ϑ on the torus (Fig. 1a). In this case the system dynamics is periodic, and the Poincaré mapping (Fig. 1b) represents a finite set of points on the secant plane that consecutively go into one another during iterations. When the frequencies ω_1 and ω_2 are incommensurable, i. e., ω_1/ω_2 is an irrational number, then for $t \to \infty$ the phase trajectory covers the entire torus densely. In this case the system dynamics is quasi-periodic, and there exists an invariant closed curve of the Poincaré mapping. By means of a suitable mapping this curve can be transformed into a circle. Hence, the system behavior can be studied with the aid of a circle mapping.

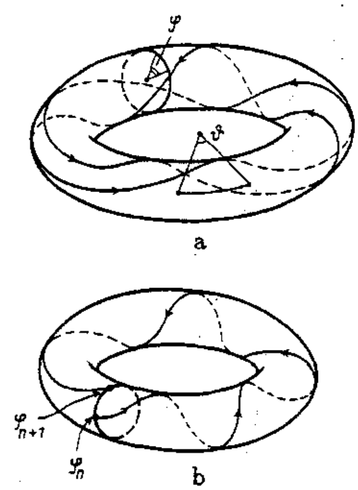

Fig. 1

Phase trajectory on a torus for $\omega_1/\omega_2 = 5/2$ (a) and Poincaré mapping as the function φ_{n+1} of φ_n (b).

Many problems, such as the periodically excited Josephson junction [24], some problems in chemical kinetics [25], etc. [26–28], lead to the study of a circle mapping.

Let us take the angle φ (Fig. 1b) as a coordinate on the torus. Then, in the general case, we have

$$\varphi_{n+1} = f(\varphi_n, \mu) = \varphi_n + g(\varphi_n, \mu), \bmod 2\pi, \tag{5}$$

in the Poincaré mapping, where μ is a parameter and $g(\cdot)$ is the shift with respect to φ. The value of φ and the function g are defined with an accuracy of 2π. If the transformation f is smooth, then g is a smooth periodic function, $g(\varphi + 2\pi, \mu) = g(\varphi, \mu)$.

The function f can either be a monotonically increasing one on the interval $[0, 2\pi]$ (one-to-one function) or it can have a maximum and a minimum (i. e., non-one-to-one function). For a monotonic function f the character of the dynamic of mapping (5) is determined by the rotation number, which can be written as

$$\rho = \lim_{n\to\infty} \frac{1}{2\pi} \frac{f^{(n)} - \varphi_0}{n}, \tag{6}$$

where $f^{(n)} = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$ is the nth iteration of the mapping. If ρ is a rational number, $\rho = p/q$, then the mapping of the circle has the same number of stable and unstable periodic orbits (cycles) of period q that wind themselves around the circle p times. Almost all the other orbits tend to stable cycles so that for a rational ρ the mapping dynamics is periodic.

When the rotation number is irrational, the orbit of almost every point turns out to be dense on the circle.

For a non-one-to-one function f, the mapping of the circle can have very complex modes of behavior. Moreover, for such a function f there can coexist two different types of chaos or a cycle and a chaos in different regions of the parameter space.

The object of our investigation will be the so-called *sine* (or standard) mapping of a circle:

$$\varphi_{n+1} = f(a, b, \varphi_n) = \varphi_n + a + b \sin \varphi_n, \bmod 2\pi, \tag{7}$$

where a and b are parameters. For $b < 1$ the function f is a C^∞-diffeomorphism. Depending on the rotation number (6), which can be rational or irrational, the mapping of the circle demonstrates periodic or quasi-periodic dynamics, respectively. For this case the mapping is rather thoroughly studied [29, 30].

For $b > 1$ the function f is no longer a diffeomorphism, and mapping (7) does not define a one-to-one transformation. Therefore for certain values of the parameters a and b mapping (7) can be in a bistable state: it can admit of different rational rotation numbers (6) depending on the initial φ_0. Moreover, as was shown numerically in [31, 32] and proved analytically in [33], for $b > 1$ and certain relationships between the parameters a and b, mapping (7) has no stable periodic trajectories of any period, i.e., it behaves chaotically and is characterized by a positive Lyapunov exponent.

Let us fix the parameter $b = \text{const} > 1$ and denote by A_c the set of values of the parameter a corresponding to the chaotic dynamics of the circle mapping (7). As the number of iterations n increases, let the parameter a vary periodically with period m according to (4a and b): $a = a(n) \equiv a_n$, $a_{m+n} = a_n$, and $a_{k+n} \neq a_n$ for $1 \leqslant k < m$. Then mapping (7) can be written in the form

$$\begin{aligned}
\varphi_{m(n+1/m)} &= \varphi_{mn} + a_1 + b \sin \varphi_{mn}, \\
\varphi_{m(n+2/m)} &= \varphi_{m(n+1/m)} + a_2 + b \sin \varphi_{m(n+1/m)}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\varphi_{m(n+1)} &= \varphi_{m(n+1-1/m)} + a_m + b \sin \varphi_{m(n+1-1/m)}, \\
\varphi_{m(n+1)+1} &= \varphi_{m(n+1)} + a_1 + b \sin \varphi_{m(n+1)}, \bmod 2\pi, \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{8}$$

Suppose that in this relation we have $a_i \in A_c$, i.e., any value of $a_1, \ldots, a_m$ in (8) corresponds to a chaotic mode of unperturbed mapping (7). In this case, in an analysis of mapping (8), one would expect that, since each "link" corresponding to a concrete value of $a_j \in A_c$, $1 \leqslant j \leqslant m$, manifests chaotic properties, mapping (8) on the whole would also have a chaotic behavior. However, this is not always the case. Among the possible values of $a_i \in A_c$, $i = 1, \ldots, m$, there are definite $a_i^d \in A_c$ for which mapping (8) generates stable cycles of finite period, i.e., destochastization takes place.

To analyze mapping (8) in detail we take, for simplicity, $m = 2$. Then the parameters $a_1, a_2 \in A_c$ form a 2-cycle, and (8) is rewritten as

$$\begin{aligned}
\varphi_{2n+1} &= \varphi_{2n} + a_1 + b \sin \varphi_{2n}, \\
\varphi_{2n+2} &= \varphi_{2n+1} + a_2 + b \sin \varphi_{2n+1}, \bmod 2\pi.
\end{aligned} \tag{9}$$

As a criterion for chaotic or regular behavior of the mappings can serve the Lyapunov exponent calculated by the formula

$$\Lambda(x_0) = \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \ln \left| \frac{df}{dx} \right|_{x_i}. \tag{10}$$

If $\Lambda(x_0) > 0$, then for the given x_0 the mapping generates nonperiodic sequences that do not tend to periodic ones with increasing number n of iterations. In the case $\Lambda(x_0) < 0$, the mapping has a periodic dynamics.

Prior to analyzing mapping (9) and determining the destochastization parameters $a_{1,2}^d \in A_c$, $b = \text{const} > 1$, it is necessary to take into account several important arguments. First assume that circle mapping (7) has a stable k-cycle for $a = a_1$ and a stable l-cycle for $a = a_2$ (for the same value of $b = \text{const}$). Recall that for the numerical observation of a cycle of any multiplicity it is necessary to perform preliminary iterations (whose number can sometimes be quite large) to avoid transient processes. Denote by $N_{\min}$ the minimum number of iterations after which the mapping attains the cycle for fixed parameter values. If the initial φ_0 is chosen so that the iterations fall on the cycle at once, then we have $N_{\min} = 0$. However, it is practically impossible to choose such initial conditions for a complicated cycle, and the probability of random attainment of the cycle is equal to zero. Consequently, for φ_0 that is not specially selected, mapping (9) may demonstrate highly diverse types of behavior because, generally, the minimum numbers of iterations for mapping (7) are different: $N_{\min}^{(1)}$ for $a = a_1$ and $N_{\min}^{(2)}$ for $a = a_2$. In this case the representing point remains sufficiently long in the transient state, so that on the whole, the dynamics of mapping (9), though regular, can be very complex in some cases.

We now assume that mapping (7) demonstrates a chaotic dynamics for both $a = a_1$ and $a = a_2$, $b = \text{const} > 1$. In this case, too, definite numbers of preliminary iterations are needed for the system to demonstrate steady behavior: $N_{\min}^{(1)}$ for $a = a_1$ and $N_{\min}^{(2)}$ for $a = a_2$. If the initial conditions are again chosen so that φ_n falls on a chaotic trajectory, then $N_{\min} = 0$. When the values of a_1 and a_2 are close, it can be expected that the regions occupied by the chaotic trajectories $\varphi_0^1, \varphi_1^1, \varphi_2^1, \dots, \varphi_n^1, \dots$ (for $a = a_1$) and $\varphi_0^2, \varphi_1^2, \varphi_2^2, \dots, \varphi_n^2, \dots$ (for $a = a_2$) overlap for $n \to \infty$. In other words, in circle mapping (7) there are coinciding values of φ_0 for which both $N_{\min}^{(1)} = 0$ and $N_{\min}^{(2)} = 0$ for $a = a_1$ and $a = a_2$, respectively.

Thus, for a correct transition from unperturbed mapping (7) to perturbed mapping (9) with the condition $a_1, a_2 \in A_c$, it is necessary to set φ_0 in (9) so that $N_{\min}^{(1)}|_{a=a_i} = 0$, $i = 1, 2$. The same refers to the more complex mapping (8) as well. The fulfilment of these conditions ensures that perturbed mapping (9) or (8) will not undergo complex transient processes corresponding to different values of $N_{\min}^{(i)}$, $i = 1, 2, \dots, m$.

For the condition $N_{\min}^{(i)}|_{a=a_i} = 0$, $i = 1, 2, \dots, m$, to be satisfied it is necessary to calculate the probability distribution $P_i(\varphi)$ for each a_i, $i = 1, 2, \dots, m$, in mapping (7) and to find the values of φ for which P_i overlap. The distribution $P(\varphi)$ characterizes the probability for the representing point of the circle mapping to fall at a given point in the interval $[0, 2\pi]$ for $n \to \infty$. For stable k-cycles the distribution $P(\varphi)$ is a superposition of delta functions located at the fixed points corresponding to this cycle. In the case of chaotic dynamics the distribution $P(\varphi)$ may have a finite value in some intervals of values of φ within the range $[0, 2\pi]$.

We now come back to mapping (9). The easiest way to make the condition $a_1, a_2 \in A_c$ hold is to select a discrete step Δa with respect to the parameter $a, b = \text{const} > 1$ and to determine *discrete* intervals $D_i(\Delta a, b)$ where mapping (7) demonstrates chaotic behavior. For example, for $b = 2.0$, $\Delta a = 10^{-3}$, and $\varphi_0 = 5.0$ these intervals in the region $2 \leqslant a \leqslant 5$ are $D_1 = [2.001; 2.033]$, $D_2 = [2.035]$, $D_3 = [2.037]$, $D_4 = [2.039; 2.062]$, $D_5 = [2.064; 2.109]$, $D_6 = [2.111; 2.135]$, $D_7 = [2.137; 2.158]$, $D_8 = [2.160; 2.178]$, $D_9 = [2.180]$, $D_{10} = [2.182; 2.241]$ $D_{11} = [2.243; 2.279]$, and $D_{12} = [2.281; 2.342]$. Next, for each of the points belonging to the chosen interval (or intervals) $D_j(\Delta a, b)$ it is necessary to calculate the probability distributions $P(\varphi)$, to determine their overlap regions with respect to φ, to choose the initial value φ_0 for mapping (9), and only after that to find the parameters a_1^d, $a_2^d \in A_c$, for which $\Lambda(\varphi_0) < 0$ in (9). It is easy to see that the suggested method of search for destochastization parameters $a_{1,2}^d$ is very laborious. And in the case we have $m > 2$ in mapping (8), it becomes inoperative at all. Nevertheless, for mapping (9) we managed to determine numerically the set of values of a_1^d, $a_2^d \in A_c$ for which the mapping generates stable cycles of finite period.

The stable cycle of shortest period (equal to 6) in mapping (9) appears for $b = 2.3$, $a_1^d = 2.832$, $a_2^d = 2.833$, and $\varphi_0 = 5.68$. It is formed by the following fixed points: $\varphi_1^* = 0.4430$, $\varphi_2^* = 0.8887$, $\varphi_3^* = 4.2610$, $\varphi_4^* = 5.0244$, $\varphi_5^* = 5.5061$, and $\varphi_6^* = 5.6674$. The probability distribution $P(\varphi)$ for the indicated values of the parameters a_1, a_2, and b are shown in Fig. 2 a and b, where it is seen that $P(\varphi)|_{a_1}$ and $P(\varphi)|_{a_2}$ have much in common and overlap with respect to φ in several places.

For $b = 2.0$, $\Delta a = 10^{-3}$, and the discrete interval $D(\Delta a, b) = D_1 \bigcup D_2 \bigcup D_3 \bigcup D_4$ of chaotic behavior (see above), the destochastization regions of mapping (9) in the space of parameters (a_1, a_2) are shown in Fig. 3.

For $m > 2$ the investigation of mapping (8) becomes much more complicated. However, the destochastization phenomenon takes place in this case as well. We managed to reveal the possibility of chaos suppression up to the value of $m = 8$. A further increase of m encounters substantial technical difficulties and therefore was not carried out.

Thus, a definite variation of an *additive* parameter (that can be regarded not only as a parametric

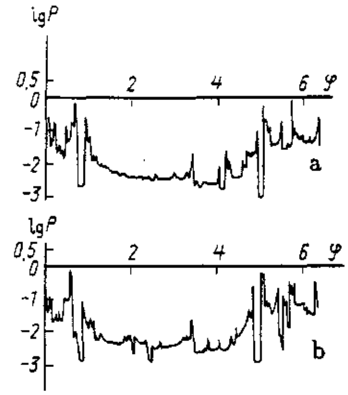

Fig. 2

Probability distribution for mapping (7): $b = 2.3$; $a = 2.832$ (a) and $a = 2.833$ (b).

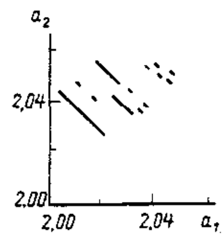

Fig. 3

Regions of destochastization of mapping (9) in the interval $D = D_1 \bigcup D_2 \bigcup D_3 \bigcup D_4$.

action but sometimes as a force action (see (4a)) results in degeneration of the chaotic behavior and in system transition to the mode of stable regular oscillations. We now discuss the case of multiplicative action on the system, which corresponds to the above mentioned mapping (4b).

Let b be the control parameter, $b = b(n) \equiv b_n$, and the parameter a remain constant in the circle mapping (7). Then it is necessary to set $a_1 = a_2 = \cdots = a_m = a$, $b = b_n > 1$, $b_{m+n} = b_n$, and $b_{k+n} \neq b_n$ for $1 \leqslant k < m$ in mapping (8). However, as follows from the numerical analysis, these changes will not affect the main conclusions. Among the various values of $b_i \in B_c$, $i = 1, \ldots, m$, $a = \text{const}$, for which circle mapping (7) shows chaotic behavior, there are definite values $b_i^d \in B_c$ resulting in transition of the perturbed mapping (8) from the mode of steady chaotic oscillations to a periodic mode. Here B_c is the set of values of the parameter $b > 1$, $a = \text{const}$, corresponding to the chaotic dynamics of mapping (7). Following the above plan, we managed to reveal this transition for the values $2 \leqslant m \leqslant 5$. For instance, for $m = 2$, $a = 2.8$, $b_1^d = 2.447$, and $b_2^d = 2.454$, mapping (8) generates a stable 6-cycle formed by the points $\varphi_1^* = 4.291781$, $\varphi_2^* = 4.851670$, $\varphi_3^* = 5.228366$, $\varphi_4^* = 5.893849$, $\varphi_5^* = 1.481846$, and $\varphi_6^* = 0.442959$. For $m = 3$, $a = 2.8$, $b_1^d = 2.511$, $b_2^d = 2.512$, and $b_3^d = 2.513$, a 16-cycle becomes stable; for $m = 4$, $a = 2.8$, $b_1^d = 2.438$, $b_2^d = 2.439$, $b_3^d = 2.460$, and $b_4^d = 2.470$, a 12-cycle is stable, etc. Generally speaking, for the same values of m and a and different $b_1^d, \ldots, b_m^d$, mapping (8) can generate cycles of various multiplicities.

Hence, certain perturbations of the parameter that enters the system multiplicatively result in the appearance of order from chaos, i. e., in destochastization.

4. CONCLUSION

A mathematical description of physical systems always necessarily involves some approximations. Usually these approximations do not take into account small external perturbations. As a rule, it is assumed that the latter can be neglected. However, as follows from the aforesaid, this is not always legitimate. Irrespective of whether they enter the equations additively or multiplicatively, small external periodic variations of the parameters may lead to a qualitatively incorrect idea of the dynamics of the simulated system, and it is not known in advance whether these periodic perturbations should be included in the mathematical model or can be neglected. Therefore, when modeling systems with chaotic behavior, one should be very careful in interpreting the numerical calculation results.

What has been said also refers to systems in which chaos is created by an infinite chain of period-duplication bifurcations and is effectively described by means of quadratic mappings [34].

Another important aspect of the presented results is the phenomenon of self-organization. Destochastization is probably an integral element of self-organization processes in various systems when the observable order originates from a well-developed chaos. And, although we have described the parametric destochastization only for localized systems, it is probable that in distributed media an analogous situation can take place: a spatial-temporal ordering is attained. Preliminary numerical studies in this area indicate that a definite parametric action on distributed systems sometimes leads to formation of complex ordered spatial-temporal patterns.

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