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THE MINIMAL NOISE FACTOR OF SOLID-STATE GRAVITATIONAL ANTENNAS

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The minimal noise factor of a solid-state gravitational antenna with an arbitrary displacement pickup has been calculated on the basis of electromechanical transformation theory for a realistic model of external effects. A physically achievable algorithm for output signal processing is discussed.

1. There is extensive literature devoted to noise analysis and calculation of threshold sensitivity for solid-state gravitational antennas (GA) with various types of electromechanical transducers (EMT). The current state of the problem is quite thoroughly described in [1].

General methods for investigating fluctuations in experiments with test bodies based on the theory of linear systems are elaborated in [2]. In this approach, an EMT is regarded as a linear (for a weak signal) quadripole with constant parameters [3, 4]. This class of EMTs includes piezo- and electrostatic transducers [5], tunnel displacement pickups [6], etc.

The data from [2] can also be applied to GAs with parametric EMTs of the modulator-demodulator type if one uses a so-called low-frequency equivalent.

The aim of the present paper is (i) to calculate the minimal noise factor of a GA with a wide-band EMT for a desired signal $F(t)$ of finite duration $\hat{\tau}$, and (ii) to design an optimal device for processing the GA output signal at given primary noise parameters of the EMT and of the final preamplifier as a coordinate measure [2].

The investigation results make it possible to generalize the algorithm developed in [2] for calculating the threshold signal amplitude $[F_0(\hat{\tau})]_{\min}$ to more complex measurement systems, and the optimal methods of processing useful information help improve the sensitivity of the already existing experimental installations.

2. To investigate the physical processes in a GA + EMT system it is advisable to employ the principle of electromechanical analogs [5]:

$$[pM + H + K/p]U = F, \qquad U = pX,$$
$$[pL + R + (pC)^{-1}]I = V, \qquad I = pq,$$

where M, H, and K are mass, friction coefficient, and rigidity of the mechanical analog, respectively; $p = d/dt$; F, U, and X are the mechanical quantities of force, velocity, and displacement, respectively; L, R, and C are inductance, resistance, and capacitance of the electrical analog; and V, I, and q are voltage, current, and charge, respectively.

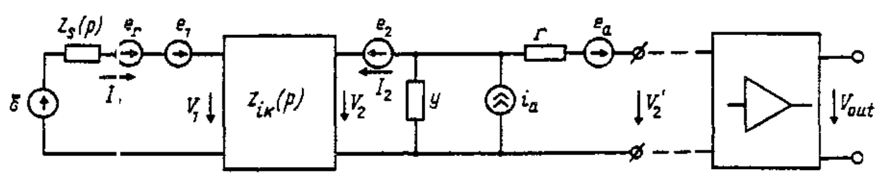

Fig. 1

An equivalent circuit for a solid-state GA with an EMT as a linear qruadripole with constant parameters (see above) is presented in Fig. 1, where the following notation is used: $Z_s(p) = pL + R + (pC)^{-1}$ is the GA impedance; $[Z_{ik}(p)]$ is the matrix of Z-parameters [3] of an EMT described by the equations of motion

$$V_1 = Z_{11}I_1 + Z_{12}I_2, \quad V_2 = Z_{21}I_1 + Z_{22}I_2;$$

$\mathcal{E}(t) = ES_0(t)\cos(\omega_s t + \varphi_0)$ is the desired signal; $S_0(t) = 1$ for $0 \leqslant t \leqslant \hat{\tau}$ and $S_0(t) \equiv 0$ for $t < 0$ and $t > \hat{\tau}$; φ_0 is the unknown initial phase; $Y(p)$ is the input conductance of the voltage preamplifier; and V_2' is the resultant signal at the preamplifier input.

The sources of fluctuations in the system are: 1) thermal noise e_R of resistance R; 2) Langevin noise sources $e_{1,2}$ of the EMT as a quadripole [4]; and 3) external fluctuations e_a and i_a of the preamplifier without internal feedback.

The energy spectra of these noise components are assumed to be known.

Neglecting the input conductance Y in the operating range $f_g \approx 10^3$ Hz, we obtain the following equations for the circuit shown in Fig. 1:

$$\begin{aligned} Z_0(p)I_1 &= \mathcal{E} + e_R + e_1 - Z_{12}(p)i_a, \\ V_2' &= Z_{21}(p)I_1 + Z_{22}(p)i_a + e_a - e_2, \end{aligned} \tag{1}$$

where $Z_0(p) = Z_s(p) + Z_{11}(p)$.

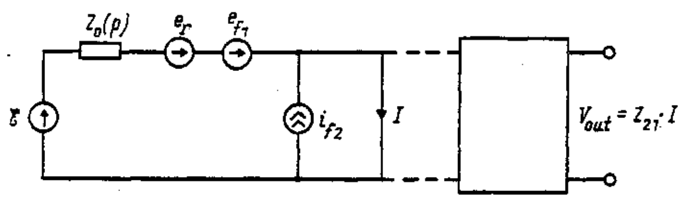

Fig. 2

The system of equations (1) makes it possible to pass to a simplified circuit demonstrated in Fig. 2, which is equivalent to the original circuit (Fig. 1) with respect to sensitivity. Here

$$e_{f1} = e_1 - Z_{12}(p)i_a, \quad i_{f2} = Z_{21}^{-1}(p)[Z_{22}(p)i_a + e_a - e_2]. \tag{2}$$

The resultant current

$$I = I_1 + i_{f2}$$

is related to the preamplifier input voltage by the formula

$$V_2' = Z_{21}(p)I.$$

The energy spectra of Langevin sources (2) determine the primary noise parameters [4] of the oscillation recording system at a given frequency $\omega > 0$:

$$\begin{aligned} N_e(\omega) &= \langle |e_{f1\omega}|^2 \rangle = N_e' + |Z_{12}(j\omega)|^2 \langle |i_{a\omega}|^2 \rangle, \\ N_1(\omega) &= \langle |i_{f2\omega}|^2 \rangle = N_i' + |Z_{21}(j\omega)|^{-2}\left[|z_{22}(j\omega)|^2 \langle |i_{a\omega}|^2 \rangle + \langle |e_{a\omega}|^2 \rangle\right], \\ N_{ei}(j\omega) &= \langle e_{f1\omega} i_{f2\omega}^* \rangle = N_{ei}' - \left[Z_{22}(j\omega)|Z_{21}(j\omega)\right]^* Z_{12}(j\omega)\langle |i_{a\omega}|^2 \rangle, \end{aligned} \tag{3}$$

where N_e', N_i', and N_{ei}' are the primary noise parameters of the EMT as a quadripole and $\langle\ \rangle$ is the statistical operator.

An analysis of (3) shows that for a reciprocal EMT ($Z_{12}(p) = Z_{21}(p)$) the external sources e_{f1} and i_{f2} are correlated even in the case when $N_{ei}' \supset \langle e_{1\omega} e_{2\omega}^* \rangle = 0$.

3. For the simplified equivalent circuit in Fig. 2 we find

$$Z_0(p)I(t) = \mathcal{E}(t) + e_n(t) = U(t), \tag{4}$$

where $e_n = e_R + e_{f1} + Z_0(p)i_{f2}$.

In a single-frequency mode for which

$$Z_0(p) = pL_0 + R_0 + (pC_0)^{-1} = L_0(p + 2\delta_0 + \omega_0^2/p),$$

the solution to Eq. (4) is sought in the form of a quasi-harmonic forced oscillation with frequency $\omega_s \approx \omega_0$. Then for a wide-band EMT we have

$$V_2' \approx A\cos(\omega_s t + \vartheta_0) - B\sin(\omega_s t + \vartheta_0).$$

The quadrature components A and B can be found from the following system of truncated equations:

$$pA - \Delta B \simeq \alpha U_1, \quad \Delta A + pB \simeq \alpha U_2, \tag{5}$$

where $\Delta = \omega_s - \omega_0$ is the possible detuning; $\alpha \approx |Z_{21}(j\omega_0)|(2L_0)^{-1}$ is a scale factor; $\vartheta_0 \approx \arg Z_{21}(j\omega_0)$; $U_{1,2}(t) = \pm 2\langle\langle U(t)(\cos\omega_s t, \sin\omega_s t)\rangle\rangle$; and $\langle\langle\ \rangle\rangle$ symbolizes averaging over the period $\tau_s = 2\pi/\omega_s$ of forced oscillations.

The energy spectra of the quadrature components $U_{1,2}(t)$ are given by the formulas

$$\begin{gathered} S_1(\omega) = S_2(\omega) = S(\omega) = \langle|U_{n\omega}[j(\omega_s+\omega)]|^2\rangle + \langle|U_{n\omega}[j(\omega_s-\omega)]|^2\rangle = 8L_0^2N_i(\delta^2+\xi^2+\omega^2); \\ S_{12}(j\omega) = j\{\langle|U_{n\omega}[j(\omega_s-\omega)]|^2\rangle - \langle|U_{n\omega}[j(\omega_s+\omega)]|^2\rangle\} \approx 16L_0^2N_i(j\omega\xi). \end{gathered} \tag{6}$$

Here

$$N_i = N_i(\omega_0); \quad \delta^2 = \delta_d^2 + (\delta_0 + \delta_e)^2; \quad \xi = \delta_m - \Delta, \tag{7}$$

where $\delta_d = (2L_0N_i)^{-1}\left[[(N_R + N_e)N_i - |N_{ei}|^2]_{\omega=\omega_s}\right]^{1/2}$, $N_R = \langle|e_{R\omega}|^2\rangle$, and $\delta_{e,m} = (2L_0N_i)^{-1}\times [\mathrm{Re}\,N_{ei},\ \mathrm{Im}\,N_{ei}]_{\omega=\omega_0}$.

An analysis of (6) shows that for $\xi \neq 0$ the noise in the quadrature components turns out to be correlated at non-coincident instants of time. Therefore to design an optimal receiver it is necessary to use the general methods of processing useful information in multichannel systems with correlated noise [7].

4. It can be shown [7] that for a deterministic signal ($\varphi_0 = 0$) an optimal receiver consists of two series-connected units. The first of them is a two-channel inversion filter whose structure is determined by system of equations (5). The other unit is an optimal device for processing the low-frequency oscillations $U_{1,2}$ which forms the functional [7]

$$H = K_0 \sum_{i=1,2} \int_{-\infty}^{\infty} U_i(t) r_i(t)\, dt, \tag{8}$$

where K_0 is an arbitrary factor and the spectra of the weight functions $r_i(t)$ are given by the formulas

$$r_{\omega\,1,2} = S_{0\omega}[S(\omega) - S_{12}(j\omega)]/[S^2(\omega) - \mathrm{Im}^2 S_{12}(j\omega)]. \tag{9}$$

The correlation receiver, whose structure is determined by expression (8), can be replaced by a passive two-channel circuit having pulse responses of the linear filters in its two channels $H_{1,2}(t) = K_0 r_{1,2}(t_0 - t)$, where t_0 is the observation time.

Whence, taking into account formulas (5), (8), and (9), we find the Fourier spectrum of the output signal of the optimal two-channel filter:

$$\begin{aligned} H_{0\omega} = K_0 \exp\{-j\omega t_0\} \sum_{i=1,2} U_{\omega i} r_{\omega i}^* = K_0[1 - \exp\{-j\omega\hat{\tau}\}\times \\ \times \exp\{-j\omega(t_0 - \hat{\tau})\}[S^2(\omega) - \mathrm{Im}^2 S_{12}(j\omega)]^{-1}(\eta_{a\omega}A_\omega + \eta_{b\omega}B_\omega)]. \end{aligned} \tag{10}$$

In (10) the following notation is used:

$$\begin{aligned} \eta_{a\omega} = \eta_{a\omega}(j\omega) = S(\omega) - \Delta(j\omega)^{-1}S_{12}(j\omega), \\ \eta_{b\omega} = \eta_{b\omega}(j\omega) = \Delta(j\omega)^{-1}S(\omega) + S_{12}(j\omega). \end{aligned} \tag{11}$$

Formulas (10) and (11) describe an optimal algorithm of processing the quadrature components A and B of the GA output voltage under a deterministic action ($\varphi_0 = 0$). The physical feasibility of the algorithm is ensured by the choice of the necessary delay t_0.

The signal-to-noise ratio ρ of the optimal receiver (10) is given by the formula [7]

$$\rho = (2\pi)^{-1}E^2 \int_{-\infty}^{\infty} d\omega\, |S_{0\omega}|^2 S(\omega)/[S^2(\omega) - \mathrm{Im}^2 S_{12}(j\omega)] = \rho_0 f^{-1}, \tag{12}$$

where $\rho_0 = (E^2\hat{\tau}/2)N_R$ is the signal-to-noise ratio for an ideal registration system and f is the EMT noise factor:

$$f = 4L_0^2(N_i/N_R)_{\omega_0}(\delta^2 + \xi^2)\delta\hat{\tau}\gamma(\delta\hat{\tau}, \xi). \tag{13}$$

Here

$$\gamma(x, \xi) = \left[x + (\delta^2 + \xi^2)^{-1}(\delta^2 - \xi^2)[\exp\{-x\}\cos\xi\hat{\tau} - 1] - 2\xi\delta(\delta^2 + \xi^2)^{-1}\sin\xi\hat{\tau}\right]. \tag{14}$$

Formulas (13) and (14) make it possible to determine the GA signal-to-noise ratio (12) for known values of the EMT and preamplifier primary noise parameters (4) and a desirable signal of finite duration $\hat{\tau}$ and a possible detuning $\Delta = \omega_0 - \omega_s$.

In the particular case of reception of ultrashort pulses ($\hat{\tau} \to 0$) with uniform spectrum and $N_R \to 0$ we have

$$\rho \approx (2/\hat{\tau})^{-1}E\left[L_0(N_e N_i - \mathrm{Im}^2 N_{12})_{\omega=\omega_0}\right]^{-1/2},$$

which coincides with the result presented in [2].

5. An analysis of the basic formula (13) shows that for a given duration $\hat{\tau}$ of the external action the noise factor f attains a minimum for an optimal detuning $\Delta_{\mathrm{opt}} = \delta_m$. In this case we have $\xi = 0$ (see (6) and (7)), and, consequently,

$$f_{\min} = 4L_0^2(N_i/N_R)\delta^2 x[\exp\{-x\} + x - 1]^{-1}, \tag{15}$$

where $x = \delta\hat{\tau}$.

However, for $\xi = 0$ we have $S(\omega) \propto (\omega^2 + \delta^2)$, $S_{12}(i\omega) = 0$. Therefore for the optimal detuning the processing algorithm (10) is substantially simplified ($\varphi_0 = 0$):

$$H_{\mathrm{opt},1} = [1 - \exp\{-j\omega\hat{\tau}\}]K_{\mathrm{opt}}(j\omega)[A_\omega + \delta_m(j\omega)^{-1}B_\omega]. \tag{16}$$

Here

$$K_{\mathrm{opt}}(j\omega) = K_0\frac{1}{(j\omega + \delta)}\left[\frac{1}{j\omega + \delta}\exp\{j\omega(t_0 - \hat{\tau})\}\right]^*. \tag{17}$$

The optimal linear filter (17) can have the form of a physically feasible whitening filter [8] and, in the general case, a physically not feasible filter matched with the first unit output signal, which are connected in series. However, for $\delta(t_0 - \hat{\tau}) \gg 1$ the employment of a physically feasible matched filter with the pulse response

$$h(t) \propto \exp\{\delta t\}, \quad 0 < t < t_0, \tag{18}$$

does not result in notable worsening of noise immunity of the reception. Naturally, practical realization of a linear device with pulse response (18) is a rather complicated technological task.

To detect a quasi-deterministic signal with unknown initial phase, an optimal receiver must contain an additional channel corresponding to $\varphi_0 = \pi/2$ and forming (see (16))

$$H_{\mathrm{opt},2} = K_0[1 - \exp\{-j\omega\hat{\tau}\}]K_{\mathrm{opt}}(j\omega)[B_\omega - \delta_m(j\omega)^{-1}A_\omega]. \tag{19}$$

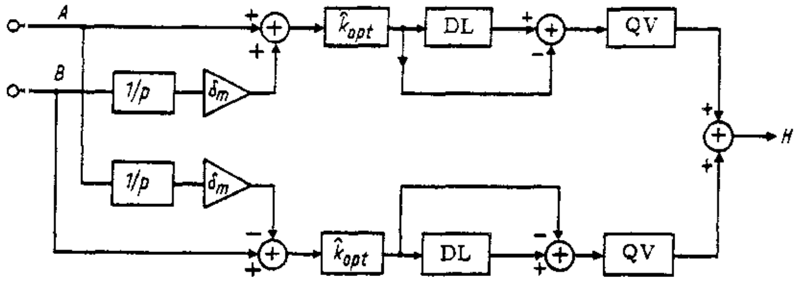

Fig. 3

The block diagram of such an optimal device is shown in Fig. 3 where the following notation is used: "DL" is the delay line, $\tau_d = \hat{\tau}$ is the delay time, and "QV" means non-linear non-inertial elements with quadratic volt-ampere characteristic $I = aV^2$ (a is a scale factor).

6. A quasi-optimal algorithm of signal processing in experiment [1] is obtained from (16), (19) for $\delta_m = 0$. Thus, the quasi-optimal filtration does not use the possibility of increasing the signal-to-noise ratio owing to noise cross-correlation in low-frequency oscillations $U_{1,2}(t)$ at non-coincident instants of time. The gain in the signal-to-noise ratio for the optimal algorithm (16), (19) as compared to the quasi-optimal one is determined by the formula (see (13))

$$q = (\delta_Q/\delta)^3 f(\delta_Q\hat{\tau}, 0)/f(\delta\hat{\tau}, 0), \tag{20}$$

where $\delta_Q^2 = \delta^2 + \delta_m^2$.

In the two limiting situations $\delta\hat{\tau} < \delta^*\hat{\tau} \ll 1$ and $\delta^*\hat{\tau} > \delta\hat{\tau} \gg 1$, the coefficient q is approximately expressed as

$$q_1 \approx \left[1 + (\delta_m/\delta)^2\right]^{1/2}, \qquad q_2 = \left[1 + (\delta_m/\delta)^2\right]^2.$$

MAIN RESULTS

For given primary noise parameters of a wide-band EMT and a voltage preamplifier as a coordinate meter, the minimal noise factor of the registration system is calculated for a desired signal of finite duration $\hat{\tau}$ at possible noncoincidence of the external action frequency ω_s and the input circuit resonant frequency ω_0.

An optimal algorithm for processing useful information has been elaborated for a GA with a wide-band EMT, which takes into account the noise cross-correlation in the quadrature components of the GA output signal at noncoincident instants of time.

Example. Passive piezoelectric transducers [1, 9], for which

$$Z_{12}(p) = Z_{21}(p) = K_{e\mu}(pC_t)^{-1}, \quad Z_{22}(p) = (pC_t)^{-1},$$

(where $K_{e\mu}$ is the electromechanical coupling coefficient and C_t is the transducer capacitance in the absence of displacements), have some advantages in the continuous tracking mode. The primary noise parameters N_e', N_i', and N_{ei}' of the piezoelectric transducer were determined in [9]. In view of (4), we have

$$N_e - N_e' = K_{e\mu}^2(\omega_0 C_t)^2 \langle |i_{a\omega}|^2\rangle_{\omega_0},$$
$$N_i - N_i' = (\omega_0 C_t)^2 K_{e\mu}^2 \left[(\omega_0 C_t)^2 \langle |i_{a\omega}|^2\rangle + \langle |e_{a\omega}|^2\rangle\right]_{\omega_0},$$
$$N_{ei} = (j\omega_0 C_t)^{-1} \langle |i_{a\omega}|^2\rangle.$$

Hence, for such an EMT

$$\delta_m = \left[2L_0 N_i(\omega_0)\omega_0 C_t\right]^{-1} \langle |i_{a\omega}|^2\rangle_{\omega_0} \neq 0, \quad \delta_e = 0.$$

Consequently, the application of the optimal output signal processing algorithm (see Fig. 3) improves the sensitivity of solid-state GAs with piezoeffect-based EMTs. For a characteristic duration $\hat{\tau} \approx 1$ the gain factor can be estimated using formula (20).

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