NONLINEAR SCHRÖDINGER-TYPE EQUATIONS
AND NONSPREADING PACKAGES


A new quadrature method for solving nonlinear Schrödinger-type wave equations is suggested. The solutions obtained have the form of nonspreading wave packages. Application of the method made it possible for the first time to solve the generalized nonlinear Schrödinger-type equation.

1. A LINEAR SCHRÖDINGER-TYPE EQUATION AND SPREADING OF A PACKAGE

Nonlinear Schrödinger-type equations are used to describe the propagation of wave packages in dispersing physical media. This work suggests a method for obtaining nonlinear equations having solutions in the form of a package.

Consider the one-dimensional equation (Schrödinger equation)

$$i \frac{\partial \Psi}{\partial t} = -\frac{\partial^2 \Psi}{\partial x^2},$$

whose solution can be written in the form

$$\Psi(x, t) = \frac{1}{(2\pi)^{1/2}} \int \tilde{\Psi}(k) \exp\{i(kx - \omega(k)t)\} \, dk.$$

For a wave package, the $\tilde{\Psi}(k)$ function must have a local maximum at some point $k = p$. The dispersion law is determined by the dependence $\omega(k) = k^2$ and is nonlinear. A wave package having finite width at the initial time $t = 0$ always spreads. The derivative of the frequency with respect to the wave number determines the group velocity:

$$\frac{d\omega(k)}{dk} \bigg|_{k=p} = V.$$

The decomposition of the frequency $\omega(k)$ in the neighborhood of $k \approx p$ allows us to trace the influence of the functional dependence $\omega(k)$ on the dispersion of the wave package:

$$\omega(k) = \omega(p) + \frac{d\omega(k)}{dk} \bigg|_{k=p} (k - p) + \frac{1}{2} \frac{d^2\omega(k)}{dk^2} \bigg|_{k=p} (k - p)^2 + \ldots.$$

The sum of the series terms starting from the third one leads to package spreading. The spreading is only absent when the dispersion law is linear. This is valid not only for the Schrödinger equation, but also for wave equations which include the second derivative with respect to time, such as the equations of classic electrodynamics in the vacuum, where the linear dispersion law also implies the absence of spreading.
2. CONSTRUCTING A NONLINEAR EQUATION THAT HAS A SOLUTION IN THE FORM OF A NONSPREADING WAVE PACKAGE

Any function of one variable \( z = x - Vt \) decreasing at infinity describes a nonspreading wave package moving at the velocity \( V \) in a positive direction. By analogy with the linear theory (nonrelativistic quantum mechanics), where free motion corresponds to the \( \Psi \sim \exp\{ipz\} \) dependence of the function on the phase, we select a solution in the form

\[
\tilde{\Psi}(x, t) = y(z) \exp\{ipz\}.
\]

This function describes a nonspreading package, because both the amplitude and the phase of the function \( \tilde{\Psi} \) depend on only one variable \( z \).

If the quantities under consideration are described by \( |\Psi|^2 \) rather than \( \Psi \), the notion of a wave package can be generalized. A solution \( \Psi(x, t) \) describing such a package, which will be called a generalized wave package, can include an additional phase multiplier \( \exp\{i\delta t\} \), where \( \delta \) depends on \( V \):

\[
\Psi(x, t) = y(z) \exp\{ipz + i\delta t\}. \tag{1}
\]

We will assume that \( y = y(z) \) is a positive real-valued function.

A generalized wave package does not spread if \( |\Psi| = |\Psi(z)| \) is a function of one variable \( z \). In the case under consideration this requirement is met automatically.

For a nonspreading wave package, we can construct the nonlinear equation

\[
-i\frac{\partial \Psi}{\partial t} = -\frac{\partial^2 \Psi}{\partial z^2} + F(|\Psi|)\Psi. \tag{2}
\]

Here \( F(|\Psi|) \) is an arbitrary real-valued function. The choice of this function is only limited by the condition of problem solvability. Equations of this form are used, e.g., in nonlinear optics [1-3]. In the special case \( F \sim |\Psi|^2 \), this equation is usually called a nonlinear Schrödinger equation.

Let us introduce the linear operator \( \hat{L} = i\partial/\partial t + \partial^2/\partial x^2 \), whose action on the function \( \Psi(x, t) \) gives the relation

\[
\hat{L}\Psi = (y_{zz} + (p^2 - \delta)y) \exp\{ipz + i\delta t\},
\]

and require that the function \( y(z) \) be a solution to the equation

\[
y_{zz} + (p^2 - \delta)y - F(y)y = 0. \tag{3}
\]

Function (1) then satisfies Eq. (2), which is linear with respect to the phase of \( \Psi \)-function (1).

Equation (3) can be considered the linear Schrödinger equation

\[
y_{zz}(z) + (E_0 - U(z))y(z) = 0
\]

with energy \( E_0 = p^2 - \delta \) and potential \( U(z) = F(y(z)) \). For a number of functions \( F(y) \), a solution to this equation can be obtained analytically. The simplest solutions are obtained when the dependence \( F = F(y) \) is exponential, i.e.,

\[
F(y) = -\beta y^{2\alpha}, \quad \alpha > 0, \beta > 0.
\]

Then there exist solutions to nonlinear equations of form (2) exponentially decreasing at infinity for which the quantities \( I_1, I_2, \) and \( I_3 \) defined by

\[
I_1 = \int |\Psi|^2 dx, \quad I_2 = -\frac{i}{2} \int (\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi) dx, \quad I_3 = \int \mathcal{H} dx, \quad \mathcal{H} = |\Psi_x|^2 + 2 \int_0^{|\Psi|} F(y)y dy
\]

do not depend on time.
3. SOLVING A NONLINEAR SCHröDINGER EQUATION

We will seek a solution to Eq. (2) with nonlinearity determined by a real-valued function $F$ depending on $|\Psi|$ in the form of generalized wave package (1). The solution can be written as

$$\Psi(x, t) = y(z) \exp\{i(px - Et)\},$$

where

$$E = 2p^2 - \delta = p^2 + E_0.$$  

The $y(z)$ function satisfies the equation

$$y_{zz} + (E_0 - F(y))y = 0,$$

for which $y_z$ is an integrating multiplier [4].

The first integral of the equation has the form

$$(y_z)^2 = -E_0y^2 + V(y), \quad \text{where} \quad V(y) = 2 \int_0^y F(y')y' dy'.$$

The integration constant in this equation is selected such that

$$y(z_0) \to 0 \quad \text{and} \quad y_z(z_0) \to 0 \quad \text{as} \quad z_0 \to -\infty.$$  

The further integration is easy to perform:

$$z - z_0 = \int_{y(z_0)}^{y(z)} \frac{dy'}{\sqrt{-E_0y'^2 + V(y')}}.$$

The expression obtained is an implicit form of a solution to a nonlinear equation for any admissible function $F(y)$. We select the integration constant $z_0$ in this formula such that the solution be everywhere regular.

As an example, let us construct a solution for an equation with nonlinearity $F(y) = -\beta y^{2\alpha}$. Then the quadrature has the form

$$z - z_0 = \int_{y(z_0)}^{y(z)} \frac{dy'}{\sqrt{-E_0 - \frac{\beta}{\alpha + 1}(y')^{2\alpha}}}.$$  

A real solution exists if $E_0 < 0$. Because the boundary condition was specified for $z \to -\infty$, the integrals are taken with signs plus.

Let us evaluate this integral:

$$2\alpha\gamma(z - z_0) = \ln \left( \frac{1 - \sqrt{1 - x}}{1 + \sqrt{1 - x}} \right) |_{x_0},$$

where

$$x = \frac{\beta}{(\alpha + 1)\gamma}2^2.$$  

Select the constant $z_0$ as

$$z_0 = \frac{1}{2\alpha\gamma} \ln \left( \frac{1 - \sqrt{1 - x_0}}{1 + \sqrt{1 - x_0}} \right).$$

This reduces the divergence for $z_0 \to -\infty$, and the solution takes the form

$$z = \cosh^{-2}(\alpha\gamma x),$$  

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Fig. 1
Dependences of $a$ and $b$ on $\delta$ for $\beta = \gamma = 1$ and $\alpha = 1$ (dotted line) and 2 (solid line).

whence $y = \left\{ \frac{(\alpha + 1)\gamma^2}{\beta} \right\}^{1/\alpha} \cosh^{-1} (\alpha \gamma \zeta)$. Thus, a solution to the nonlinear Schrödinger equation

$$\frac{i}{\partial t} \frac{\partial \Psi}{\partial t} = -\frac{\partial^2 \Psi}{\partial x^2} - \beta |\Psi|^{2\alpha} \Psi$$

is the function

$$\Psi(x, t) = \left\{ \frac{(\alpha + 1)\gamma^2}{\beta} \right\}^{1/\alpha} \cosh^{-1} (\alpha \gamma (x - Vt)) \exp\{i(px - Et)\}$$

describing a nonspreading wave package with $E = p^2 - \gamma^2$. The conserved quantities are given by

$$I_1 = \frac{2^{2/\alpha - 1}}{\alpha \gamma} \left\{ \frac{(\alpha + 1)\gamma^2}{\beta} \right\}^{1/\alpha} B\left(\frac{1}{\alpha}, \frac{1}{\alpha}\right), \quad I_2 = pI_1,$$

$$I_3 = \left\{ \frac{p^2 + \alpha - 2}{\alpha + 2 \gamma^2} \right\} I_1.$$

Solutions for other functions $F(y)$ are constructed similarly. For example, the applied quadrature method makes it possible to find a solution to the more general nonlinear equation

$$\frac{i}{\partial t} \frac{\partial \Psi}{\partial x} = -\frac{\partial^2 \Psi}{\partial x^2} - \beta |\Psi|^{2\alpha} \Psi - \delta |\Psi|^\alpha \Psi,$$

which contains two nonlinearities. A solution to this equation in the form of a generalized wave package is given by

$$\hat{L} \Psi = \{y_{xx} + (p^2 - \delta)y\} \exp\{ipx + i\delta t\} \Psi(x, t) = \frac{a \exp\{i(px - Et)\}}{1 \cosh \alpha \gamma (x - Vt)}$$

where

$$E = p^2 - \gamma^2,$$

$$a = \left\{ \frac{(\alpha + 2)\gamma^2}{\delta} \right\}^{1/\alpha} \left\{ 1 + \frac{(\alpha + 2)^2 \beta \gamma^2}{(\alpha + 1) \delta^2} \right\}^{-1/2},$$

$$b = \left\{ 1 + \frac{(\alpha + 2)^2 \beta \gamma^2}{(\alpha + 1) \delta^2} \right\}^{-1/2}.$$

The dependence of $a$ and $b$ on $\delta$ for $\beta = \gamma = 1$ is shown in Fig. 1 for two $\alpha$.

The suggested method for constructing solutions in the form of nonspreading packages makes it possible to obtain solutions when the corresponding inverse scattering problem is unknown or does not exist [5].
REFERENCES


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