CAN AN ILL-POSED PROBLEM BE SOLVED IF THE DATA ERROR IS UNKNOWN?

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It is shown that finding a stable solution to an operator equation without knowledge of the error involved in its data is only possible for well-posed problems.

This paper has been prompted by the appearance of a series of works [1-4] aimed at solving ill-posed problems without taking account of the errors of their data. Is it at all possible to solve an ill-posed problem when data errors are unknown? This is a rather old problem in the theory of ill-posed problems. It arose as long ago as the early 1960s, and some of its aspects were examined by the school of A. N. Tychonoff. Unfortunately, there has been no integral theoretical consideration of the problem in the literature. For this reason, we doubt that all results pertaining to this question are known to scientists not specializing in ill-posed problems but facing the necessity of solving such problems and selecting suitable methods. We believe that a special discussion of the problem would also be useful because it may prevent further fruitless efforts (similar to those made in [1-4]) to create "practical" methods for solving ill-posed problems that would not include information on data errors. For this reason, we would like to outline the basic points of the analysis of the problem.

The key point of the analysis is answering two questions: What is an ill-posed problem, and What is its solution? To answer these questions, consider the operator equation

\[ Az = u, \quad z \in Z \]

with respect to the unknown \( z \). Many problems arising in actual practice can be written in this form. For definiteness, let the \( \mathcal{A} \) operator belong to the set \( \mathcal{L}(Z, U) \) of linear bounded operators acting from the Hilbert space \( Z \) into the Hilbert space \( U \), and let \( u \in U \).

According to the definition going back to J. Hadamard [5], problem (1) is called well-posed on the class of its "admissible" data \( \Sigma = \{(A, u)\} \) if for any data \( (A, u) \in \Sigma \), its solution \( z = z(A, u) \in Z \) exists, is unique, and continuously depends on the data (is stable). A continuous dependence means that for arbitrary "perturbed" data \( (A_h, u_\delta) \in \Sigma \) such that \( ||A_h - A|| \leq h \) and \( ||u_\delta - u|| \leq \delta \), we have the convergence \( z(A_h, u_\delta) \rightarrow z(A, u) \) as \( h, \delta \rightarrow 0 \). The numbers \( h \) and \( \delta \) are estimates of the errors of the approximate data \( (A_h, u_\delta) \) of problem (1) with exact data \( (A, u) \). If at least one of the mentioned requirements is not met, then problem (1) is called ill-posed. Typical examples of ill-posed problems are many inverse problems pertaining to processing physical experimental data, vibrational spectroscopy, astrophysics, etc.

Violation of the requirement of the existence and uniqueness of a solution to problem (1) is usually overcome by searching for some "generalized" solution to the problem. As this generalized solution, the so-called normal pseudo-solution \( \bar{z} \) (a solution in the sense of the least-squares method with a minimum norm) is often taken [6]. It exists and is unique for any admissible exact data of problem (1) from the class \( \Sigma = \{(A, u) : A \in \mathcal{L}(Z, U), u \in U, u \in R(A) \oplus R(A)^\perp \} \) and is expressed in terms of these data by the rule \( \bar{z} = A^+u \). Here \( R(A) \) and \( R(A)^\perp \) denote the ranges of the operator \( A \) and its orthogonal complement in

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regularizing algorithm that would not depend explicitly on the estimates of the errors introduced the so-called other additional assumptions (such as special spectral properties of the generalized principle of the residual suggested the regularization procedure with various techniques of selecting most interesting ones are not required.

Here, as in the effective numerical methods for solving ill-posed problems were developed based on the regularization procedure with various techniques of selecting \( \alpha \). These methods are summarized in \([16, 17]\), including computer programs.

At the same time, as early as the 1960s the question arose of whether it is possible to construct a regularizing algorithm that would not depend explicitly on the estimates of the errors \( h \) and \( \delta \) and would...
yield an approximate solution in the form $z_{h\delta} = R(A_h, u_\delta)$. This attractive possibility has first been demonstrated in a numerical experiment by Tychoffon and Glasko [18]. They suggested the so-called quasi-optimal selection of $\alpha$, where $\alpha(A_h, u_\delta)$ is found as a solution to the problem of minimizing the function

$$
\psi(\alpha) = \left\| \frac{d^2}{d\alpha^2} (A_h, u_\delta) \right\|^2, \quad \alpha \geq 0.
$$

However, all attempts to theoretically substantiate the quasi-optimal selection of $\alpha$ as a general regularizing algorithm for the general case of the infinite-dimensional Hilbert spaces $Z$ and $U$ without using additional detailed information on $\mathcal{E}$ failed. Moreover, it was shown by a number of counterexamples that the regularization procedure with a quasi-optimal selection of $\alpha$ does not give convergence of approximations to $\mathcal{E}$ for $A_h \to A$ and $u_\delta \to u$ (see, e.g., [19]). Convergence (2) of quasi-optimal approximations has only been proved for a precisely specified operator $A$ in the finite-dimensional $Z$ and $U$ [21].

A similar situation arises when $\alpha$ is selected by the general cross-validation (GCV) method [21], where $\alpha(A_h, u_\delta)$ is found as the point of the global minimum of the function

$$
G(\alpha) = \left\| (\alpha E + A_h A_h^*)^{-1} u_\delta \right\| \cdot \left\| \text{tr}(\alpha E + A_h A_h^*)^{-1} \right\|^{-1}, \quad \alpha \geq 0.
$$

Here $E$ is the unit operator in $U$.

This is not accidental. The key to understanding this phenomenon is given by the following

**Theorem.** Let $R(A_h, u_\delta)$ be a map of the set $\mathcal{L} \otimes U$ into $Z$. If $R(A_h, u_\delta)$ is a regularizing algorithm not depending explicitly on $h$ and $\delta$, then the map $P(\mathcal{A}, u) = A^+ u$ is continuous on its domain $\Sigma$.

**Proof.** Condition (2) in the definition of a regularizing algorithm implies in the equality $R(\mathcal{A}, u) = A^+ u = P(\mathcal{A}, u)$ valid for each $(\mathcal{A}, u) \in \Sigma$ and convergence $P(A_h, u_\delta) = R(A_h, u_\delta) \to A^+ u = P(\mathcal{A}, u)$ when $h, \delta \to 0$ valid for any $(A_h, u_\delta) \in \Sigma$. Therefore, the map $P(\mathcal{A}, u)$ is continuous on $\Sigma$.

A similar, but not equivalent, assertion was proved by Bakushinskii [22].

It is clear from the theorem that a regularizing algorithm not using $h$ and $\delta$ explicitly can only exist for problems (1) well-posed (stable) on the set of data $\Sigma$, which immediately follows from the continuity of the $A^+ u$ map on its domain.

The necessity of the dependence of the regularizing algorithm on data errors $h$ and $\delta$ was mentioned in the latest works by Tychoffon concerned with solving unstable systems of linear algebraic equations (see [23,24]). Some examples can also be found in [25].

Thus, according to the theorem, “all methods for solving ill-posed problems that do not use $h$ and $\delta$” are actually applicable only for solving well-posed (stable) problems. These methods include the criterion for quasi-optimality [18], the GCV method [21], a modification of the truncated SVD method described in [1,3], and some other methods. They can be applied to systems of linear algebraic equations of fixed dimensionality with nonperturbed matrix as methods for solving well-posed problems. The result formulated in the theorem explains why all attempts to theoretically substantiate the above-mentioned methods as regularizing algorithms failed in the general case of infinite-dimensional Hilbert spaces $Z$ and $U$.

Adaptation of these methods to solving ill-posed (unstable) problems is only possible if use is made of some additional detailed information on the exact solution $\mathcal{E}$, on exact data $(\mathcal{A}, u)$, or on the characteristics of the random variables $A_h$ and $u_\delta$. This information is usually present in the form of presetting the upper bound for $\|\mathcal{E}\|$ or special spectral constraints on $\mathcal{E}$. Sometimes, such information makes the problem well-posed. Theoretically, it allows us in some cases to prove convergence (2) for the regularization method with selection of $\alpha$ without using $h$ and $\delta$. This is however possible only for sufficiently “narrow” subsets of exact and approximate data of the problem. Examples of such attempts can be found in [3,4,21,26].

In our opinion, one of the recently proposed methods for selecting a regularization parameter requires special consideration. This is the so-called method of $L$-curve [2,3], which uses family (3) and in which $\alpha$ is selected as the point of maximum curvature of the $L$-curve $\{(\ln \|A_h z^2 - u_\delta\|, \ln \|z^2\|) : \alpha > 0\}$. To substantiate the method, its authors largely used model computations. This approach meets with serious objections if it is not substantiated by a theoretical proof of convergence (2). Moreover, we are sure that the method of $L$-curve is not only uncapable of solving ill-posed problems (see the theorem), but it is inapplicable to solving simplest finite-dimensional well-posed problems. This can be easily demonstrated by the example of Eq. (1) with $Z = U = R^1$, $A = I$, $u = 1$ and with the approximate data $A_h = I$ and $u_\delta = 1$ for any $h$ and
\( \delta \). Independently of \( h \) and \( \delta \), the regularization parameter selected by the \( L \)-curve method is here constant: 
\[
\alpha_L(A_h, u_h) = 1.
\]
Therefore, according to (3), the approximate solution is 
\[
\hat{\alpha} = 0.5,
\]
and it does not converge to \( \tilde{\alpha} = 1 \) as \( h, \delta \to 0 \). It is possible to construct more complex and refined examples demonstrating the invalidity of various modifications of the \( L \)-curve method.

Thus, our principal conclusion is that, generally, ill-posed problem (1) cannot be solved without using data errors \( h \) and \( \delta \), whereas there exist theoretically substantiated regularizing algorithms that generalize the quasi-optimal selection of \( \alpha \) and the truncated SVD method and explicitly use \( h \) and \( \delta \) (see [19, 27, 28]).

The results of this paper, including the theorem, remain valid for nonlinear ill-posed problems. The algorithms \( R(h, \delta, A_h, u_h) \) can also have the form of iterative methods [29].

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REFERENCES


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